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EDITED BY

FRANK MORLEY

WITH THE COOPERATION OF

A. COHEN, CHARLOTTE A. SCOTT

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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Flat-Sphere Geometry.

SECOND PAPER.

BY JOHN EIESLAND.

INTRODUCTION.

In a paper published in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV,* I have extended Lie's Line-Sphere Geometry in 3-space to an odd-dimensional space in which a special self-dual element takes the place of the line in 4-space. Conforming to the notation of this memoir, to which we shall refer in the following pages by the letter A, we denote the odd-dimensional space by S_{n-1} (n-even). The following theorem was proved (A, p. 226):

There exist in $\bar{\bar{S}}_{n-1}\infty^{\frac{n+1\cdot n+2}{2}}$ *contact-transformations which transform the* ∞^n *flat spreads*

$$x_i=ay_i+b_i,\quad z=\Sigma c_iy_i+d,\quad \{q_i=-ap_i+c_i\},\qquad i=1,2,\ldots,\frac{n-2}{2}. \quad (1)$$

into the ∞^n *spheres in* S_{n-1}. *These transformations are obtained by superposing the inverse of the generalized Lie transformation L on the contact-transformations that leave the* ∞^n *flats* (1) *invariant. These transformations have the characteristic functions:*

$$\left.\begin{array}{l} 1,\ p_i,\ y_i,\ x_i,\ q_i,\ x_ip_i+y_iq_i,\ x_ky_i-x_iy_k,\ q_kp_i-p_kq_i,\ p_ix_k+q_iy_k, \\ x_ip_k+y_iq_k,\ zy_k-y_k\Sigma q_jy_j-x_k\Sigma p_jy_j,\ zp_k-p_k\Sigma q_iy_i+q_k\Sigma p_jy_j,\ \Sigma q_jy_j, \\ \Sigma x_jq_j,\ \Sigma p_jy_j,\ 2z-\Sigma(x_jp_j+y_jq_j),\ zx_k-y_k\Sigma x_jq_j-x_k\Sigma x_jp_j,\ zq_k+p_k\Sigma x_jq_j \\ -q_k\Sigma x_jq_j,\ z^2-z\Sigma q_jy_j-z\Sigma p_jx_j+\Sigma x_jp_j\cdot\Sigma qyy-\Sigma x_jq_j\cdot\Sigma y_jp_j, \\ \hfill i,k=1,2,\ldots,\frac{n-2}{2},\ i\neq k. \end{array}\right\} \quad (2)$$

---

*"On a Flat Spread-Sphere Geometry in Odd-Dimensional Space," pp. 201–228.

The inverse of L is:

$$\left.\begin{aligned} x_i &= -(X_{2i-1}+iX_{2i})-\frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad y_i=-\frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \\ p_i &= \frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad q_i=-(X_{2i-1}-iX_{2i})-\frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, . \\ z &= \frac{\Sigma(P_{2i-1}+iP_{2i})(X_{2i-1}-iX_{2i})}{1\pm\sqrt{1+\Sigma P_i^2}}-X_{n-1}, \quad i=1, 2, \dots, \frac{n-2}{2}. \end{aligned}\right\} \quad (3)$$

The same transformations $L^{-1}G_{\frac{n+1}{2}\cdot n+2}$ transform the asymptotic curves on an $n-2$-spread M_{n-2} in $\overline{\overline{S}}_{n-1}$ into the lines of curvature on the transform of M_{n-2} in S_{n-1}. The differential equations of the asymptotic curves on M_{n-2} are

$$dx_i dp_{\frac{n-2}{2}}+dy_i dq_{\frac{n-2}{2}}=0, \quad dq_k dp_{\frac{n-2}{2}}-dp_k dq_{\frac{n-2}{2}}=0, \quad \left\{\begin{aligned} i&=1, 2, \dots, \frac{n-2}{2} \\ k&=1, 2, \dots, \frac{n-2}{2} \end{aligned}\right\}, \quad (4)$$

and those of the lines of curvature on the transform are

$$(dX_i+P_iX_{n-1})\,dP_{n-2}-(dX_{n-2}+P_{n-2}dX_{n-1})\,dP_i=0, \qquad i=1, 2, \dots, n-3.$$

I

§ 1. The lines that are here denoted as "asymptotic" are from the standpoint of the flat-sphere geometry in S_{n-1} the analogues of asymptotic lines on a surface in 3-space, inasmuch as through any point on the surface there pass $n-2$ such lines, and to these correspond by the generalized Lie transformation (3) the lines of curvature on the transform.*

The lines of curvature on a spread M_{n-2} in S_{n-1} are not necessarily *coordinate lines* of curvature in the Darboux sense. In fact, as Darboux has shown,† an $n-2$-spread has coordinate lines of curvature if and only if an $n-1$-tuple orthogonal system exists of which the given M_{n-2} is one of the $n-1$ mutually orthogonal surfaces. If a surface has coordinate lines of curvature

** I. e., in general;* in the case of special surfaces we may have "bands" of curvature or curvature $=$spreads, in which case the transform has "bands" of asymptotic curves. Such cases we shall meet with in the present paper.

† Darboux, "Leçons sur les systèmes orthogonaux et les coordinées curvilignes," pp. 133–137 and 176–182. See also a note in *Comptes Rendus*, Vol. CXXVIII, pp. 284–285, entitled "Sur les systèmes orthogonaux," by A. Pellet.

it may be shown that the coordinates X_i and the direction cosines of the normal c_i must satisfy the generalized Olinde Rodrigue formulae

$$\frac{\partial X_i}{\partial \rho_k} + R_k \frac{\partial c_i}{\partial \rho_k} = 0, \qquad \begin{array}{l} i=1, 2, \ldots, n-1, \\ k=1, 2, \ldots, n-2, \end{array}$$

that is, on a surface having coordinate lines of curvature the functions $\rho_1, \ldots, \rho_{n-2}$ are such that on a line of curvature ρ_k the function ρ_k only varies while the others remain constant. For 3-space this is always true, while for $n>3$ this condition imposes restrictions on the surface as an individual; while every surface in n-space has lines of curvature, not every surface is a member of an n-tuple orthogonal system of surfaces.

We shall say that a spread M_{n-2} in $\overline{\overline{S}}_{n-1}$ has *coordinate asymptotic lines* whenever its transform by the transformation (3) has coordinate lines of curvature.

Through the researches of Darboux and others we now know a great many orthogonal systems in n-space, and therefore also an indefinite number of surfaces having coordinate lines of curvature. If in an odd space we transform these by (3) we obtain surfaces on which the corresponding lines are coordinate asymptotic lines, but since in general, to real surface-elements in S_{n-1} correspond imaginary elements in $\overline{\overline{S}}_{n-1}$, this method is not always convenient and sometimes even impractical, since the eliminations to be performed are rather complicated if the surface in S_{n-1} is to be obtained in parametric or Cartesian form. The lines of curvature being real on a real surface, the corresponding asymptotic lines will in general be imaginary if the transform is a real surface.

§ 2. We shall represent the tangent flat F_{n-2} to an n—2-spread in S_{n-1} by the following equation:

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots+i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})X_{n-2} + (1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0. \tag{5}$$

the envelope of which is the spread M_{n-2} (A, p. 208):

$$\left.\begin{aligned} X_1 &= \frac{(\alpha_1+\beta_2)}{2}X_{n-1} - \frac{1}{2}(F'_{\alpha_1}+F'_{\beta_1}), \\ X_2 &= i\frac{(\alpha_1-\beta_1)}{2}X_{n-1} + \frac{i}{2}(F'_{\alpha_1}-F'_{\beta_1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1} &= \frac{\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{aligned}\right\} \tag{6}$$

The equations (5) may therefore be considered as the tangential equation of the spread. The equations of the lines of curvature are (A, p. 204):

$$\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta_i}}{dF'_{\beta_{\frac{n-2}{2}}}},\quad \frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha_k}}{dF'_{\beta_{\frac{n-2}{2}}}},\quad \begin{array}{l} i=1,2,\ldots,\frac{n-4}{2}, \\ k=1,2,\ldots,\frac{n-2}{2}, \end{array} \tag{7}$$

which may be put in the form, writing $\bar{p}_i$ for F'_{α_i} and $\bar{q}_i$ for F'_{β_i},

$$d\alpha_i d\bar{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\bar{q}_i=0,\quad d\beta_k d\bar{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\bar{p}_k=0. \tag{8}$$

If $\rho_1,\ldots,\rho_{n-2}$ are coordinate lines of curvature we must have

$$\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\bar{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\bar{q}_i}{\partial\rho_k}=0,\qquad \frac{\partial\beta_i}{\partial\rho_k}\frac{\partial\bar{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\bar{p}_i}{\partial\rho_k}=0, \tag{9}$$

which may be written

$$\frac{\partial\bar{q}_i}{\partial\rho_k}=\lambda_k\frac{\partial\alpha_i}{\partial\rho_k},\quad \frac{\partial\bar{p}_i}{\partial\rho_k}=\lambda_k\frac{\partial\beta_i}{\partial\rho_k},\qquad \left(\begin{array}{l} i=1,\,2,\,\ldots,\frac{n-2}{2} \\ k=1,\,2,\,\ldots,\frac{n-2}{2} \end{array}\right), \tag{10}$$

to which must also be added the $n-2$ conditions

$$\frac{\partial F}{\partial\rho_k}=\sum\left[\bar{p}_i\frac{\partial\alpha_i}{\partial\rho_k}+\bar{q}_i\frac{\partial\beta_i}{\partial\rho_k}\right]. \tag{11}$$

Expressing the conditions for the integrability of (10) we obtain the following system of $\frac{(n-2)(n-3)}{2}$ partial differential equations which must be satisfied by the α's and β's:

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0. \tag{12}$$

In the same way, expressing the condition for the integrability of (11) we are led to the system of $\frac{n-2\cdot n-3}{2}$ equations:

$$\sum\left[\frac{\partial\bar{p}_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\frac{\partial\bar{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial p_i}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}-\frac{\partial\bar{q}_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0,$$

which by means of (10) may be reduced to the following:

$$(\lambda_k-\lambda_{k'})\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{13}$$

Assuming that $\lambda_k \neq \lambda_{k'}$ for all values of k and k' * these conditions are

$$\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{14}$$

If, therefore, the lines $\rho_1, \rho_2, \ldots, \rho_{n-2}$ are coordinate lines of curvature on M_{n-2}, the functions α_i and β_i must satisfy the conditions (12) and (13′). These conditions being satisfied, we shall show that F is also a solution of (12). We have from (11),

$$\frac{\partial^2 F}{\partial\rho_{k'}\partial\rho_k}=\sum\frac{\partial p_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\sum\frac{\partial\bar{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}$$

$$=\lambda_{k'}\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}},$$

hence

$$(\lambda_k-\lambda_{k'})\frac{\partial^2 F}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}}=\lambda_{k'}(\lambda_k-\lambda_{k'})\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)$$

$$+\sum\bar{p}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right]$$

$$+\sum\bar{q}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right].$$

Now, if the condition of integrability (12) and (14) are satisfied, the right side of this equation vanishes identically; hence F is a solution of (12) *q. e. d.* It may be worth while to note that the system (12) is also satisfied by the function $\Sigma\alpha_i\beta_i$, α_i and β_i being a set of particular solutions. We have then the following:

THEOREM I. *If on a surface* (6) *the lines of curvature are coordinate lines, the curvilinear coordinates* α_i, β_i *must satisfy the system of differential equations,*

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0, \tag{12}$$

and also the $\dfrac{n-2\cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)=0. \tag{14}$$

---

*This amounts to assuming that the surface (6) is an $n-2$-spread. If $\lambda_k=\lambda_{k'}$ for *all* values of k and k', the surface is one-dimensional or a curve.

We shall transform the coordinates α_i and β_i putting

$$\left.\begin{aligned}&\alpha_1+\beta_1=2y_1, \quad \alpha_2+\beta_2=2y_3, \;\ldots\ldots, \alpha_{\frac{n-2}{2}}+\beta_{\frac{n-2}{2}}=2y_{n-3},\\ &i(\alpha_1-\beta_1)=2y_2, \quad i(\alpha_2-\beta_2)=2y_4, \;\ldots\ldots, i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})=2y_{n-2}.\end{aligned}\right\} \quad (16)$$

The equations (10) now become

$$\frac{1}{2}\frac{\partial}{\partial\rho_k}(\bar{p}_i+\bar{q}_i)=\lambda_k\frac{\partial y_{2i-1}}{\partial\rho_k}, \quad \frac{i}{2}\frac{\partial}{\partial\rho_k}(\bar{q}_i-\bar{p}_i)=\lambda_k\frac{\partial y_{2i}}{\partial\rho_k}, \quad i=1,2,\ldots\ldots,\frac{n-2}{2}, \quad (17)$$

and the new coordinates y_i are still solutions of (12). The conditions (14) are

$$\sum\frac{\partial y_i}{\partial\rho_k}\frac{\partial y_i}{\partial\rho_{k'}}=0, \qquad k,k'=1,2,\ldots\ldots,n-2, \; k\neq k'. \quad (18)$$

We write (12) in the form

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial\theta_k}{\partial\rho_{k'}}=0, \quad (19)$$

where the H's are defined by the equations

$$\left.\begin{aligned}\frac{\partial\lambda_k}{\partial\rho_{k'}}&=-(\lambda_k-\lambda_{k'})\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}},\\ \frac{\partial\lambda_{k'}}{\partial\rho_k}&=\quad(\lambda_k-\lambda_{k'})\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}.\end{aligned}\right\} \quad (20)$$

Observing that (18) expresses the $\frac{n-2\,.\,n-3}{2}$ conditions that the y's, considered as functions of the ρ's, shall form a completely orthogonal system in S_{n-2}, we put

$$dy_1^2+dy_2^2+\ldots\ldots+dy_{n-2}^2=H_1^2d\rho_1^2+H_2^2d\rho_2^2+\ldots\ldots+H_{n-2}^2d\rho_{n-2}^2 \quad (21)$$

where $H_1,\ldots\ldots,H_{n-2}$ satisfy the $\frac{n-2\cdot n-3\cdot n-4}{2}$ conditions (Darboux, *loc. cit.*, p. 165).

$$f_{kk'}(H_{k''kk'})=\frac{\partial^2 H_{k''}}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial H_{k''}}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial H_{k''}}{\partial\rho_{k'}}=0. \quad (22)$$

$n-2$ such functions being found, the functions λ_k may be found by integrating the system (20); $\bar{p}_i$, $\bar{q}_i$ and F are then determined by quadratures from (17) and (11). The systems (20), which determine the λ's, are equivalent to a single system

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=-(\lambda_k-\lambda_{k'})\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}},$$

which admits of one and only one set of solutions $\lambda_1\ldots\ldots\lambda_{n-2}$, such that

$\lambda_k = f_k(\rho_k)$ for $\rho_k = \rho_k^0$ where ρ_k^0 is an initial value of ρ_k (Darboux, *loc. cit.*, p. 350). We have then

THEOREM II. *The determination of a surface in S_{n-1} on which the lines of curvature are coordinate lines depends on the determination of a completely orthogonal system in a space S_{n-2} of one dimension less. Such a system being given the corresponding surface is found by integrating the system*

$$\frac{\partial \lambda_k}{\partial \rho_{k'}} = -(\lambda_k - \lambda_{k'}) \frac{1}{H_k} \frac{\partial H_k}{\partial \rho_{k'}}, \tag{20'}$$

and by quadratures.

The first part of this theorem has been proved by Darboux * by a different method, starting with the generalized Olinde Rodrigue's formulae for n-space. The method used here will be of advantage in the transformation to the space $\overline{\overline{S}}_{n-2}$.

Instead of determining a system of λ's from (20′), we may take any solution whatever of (19). Let Θ be such a solution; then putting $F = \Theta$ we have at once a surface on which the ρ's are coordinate lines of curvature. In case a general solution of (19) can be found, and such solution always exists since the conditions (22) are satisfied, we obtain all the surfaces having the same spherical representation of the lines of curvature.

APPLICATIONS.

§ 3. Let $H_i = 1$. Then $\frac{\partial \lambda_k}{\partial \rho_k} = 0$, or, $\lambda_k = \phi_k''(\rho_k)$. Equations (19) are now

$$\frac{\partial^2 \theta}{\partial \rho_k \partial \rho_{k'}} = 0, \tag{23}$$

the general solution of which is

$$\theta = \phi_1 + \phi_2 + \ldots + \phi_{n-2},$$

ϕ_i being a function of ρ_i alone. Taking as particular solutions of (23) the $n-2$ independent functions $y_i = \rho_i$ we have by integrating the system (17),

$$\bar{q}_i + \bar{p}_i = 2\phi'_{2i-1}, \quad i(\bar{q}_i - \bar{p}_i) = 2\phi'_{2i}, \qquad i = 1, 2, \ldots, \frac{n-2}{2},$$

and integrating (11) we find

$$F = 2 \sum_1^{n-2} \phi_i + 2c,$$

* Darboux, *loc. cit.*, pp. 178–182.

which shows that F is a general solution of (23). The tangential equation of the surface is, therefore,

$$2\rho_1 X_1+2\rho_2 X_2+\ldots+(1-\Sigma\rho_i^2)X_{n-1}+2\Sigma\phi_i+2c=0. \tag{24}$$

Substituting the values of the α's in terms of the ρ's in (6) we have the surface

$$\left.\begin{aligned} &X_{2i-1}=-\phi'_{2i-1}+\rho_{2i-1}X_{n-1}, \quad X_{2i}=-\phi'_{2i}+\rho_{2i}X_{n-1}, \\ &X_{n-1}=\frac{2\Sigma\phi'_{2i-1}\rho_{2i-1}+2\Sigma\phi'_{2i}\rho_{2i}-2\Sigma\phi_i-2c}{1+\Sigma\rho_i^2}, \\ &\qquad\qquad i=1,2,\ldots,\frac{n-2}{2}. \end{aligned}\right\} \tag{25}$$

These equations show that the surfaces have their lines of curvature plane in all the $n-2$ systems. The spherical representation of these lines are circles passing through the point $x_1=x_2=\ldots=x_{n-2}=0$, $x_{n-1}=1$, as readily appears from (24). The focal surface has $n-2$ sheets, viz.:

$$\overline{X}_i=\rho_i\phi''_k-\phi'_i, \quad \overline{X}_{n-1}=\frac{1-\Sigma\rho_i^2}{2}\phi''_k+\Sigma\rho_i\phi'_i-\Sigma\phi_i-c, \qquad i,k=1,2,\ldots,n-2,$$

where the subscript k indicates the individual sheet.*

As a particular case let us put $\phi_{2i-1}=\dfrac{k+2}{4}\rho_{2i-1}^2$, $\phi_{2i}=\dfrac{k-2}{4}\rho_{2i}^2$, $c=\dfrac{k}{4}$. Substituting in (24) and differentiating with respect to $\rho_1,\rho_2\ldots\rho_{n-2}$ in succession, we have

$$2X_{2i-1}-2\rho_{2i-1}X_{n-1}=-(k+2)\rho_{2i-1}, \quad 2X_{2i}-2\rho_{2i}X_{n-1}=-(k-2)\rho_{2i}. \tag{27}$$

Eliminating the ρ's from (24) and (27) we have the cubic surface

$$2X_{n-1}\sum_1^{n-1}X_i^2-(k-2)\sum_1^{\frac{n-2}{2}}X_{2i-1}^2-(k+2)\sum_1^{\frac{n-2}{2}}X_{2i}^2-kX_{n-1}^2$$
$$+\frac{k^2-4}{2}X_{n-1}+\frac{k(k^2-4)}{4}=0, \tag{27'}$$

a special kind of cyclide which we shall study later on.

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*The equation of the focal surface may be obtained from equations (19) A, p. 207, by introducing the variables y, instead of the α's and β's. We obtain the following:

$$\overline{X}_i=\frac{y_i\sigma_k}{2}-\frac{p_i}{2}, \quad \overline{X}_{n-1}=\frac{1-\Sigma y_i^2}{4}\sigma_k+\frac{1}{2}\Sigma p_i y_i-\frac{F}{2}, \qquad i,k=1,2,\ldots,n-2,$$

where σ_k is a root of the equation

$$\begin{vmatrix} p_{11}-\sigma & p_{12}\ldots\ldots\ldots p_{1n-2} \\ p_{21} & p_{22}-\sigma\ldots\ldots p_{2n-2} \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ p_{n-21} & p_{n-22}\ldots\ldots p_{n-2n-2}-\sigma \end{vmatrix}=0. \tag{26}$$

Let H be of the form $\frac{1}{h}$, h being a function of the ρ's. Darboux has shown* that h must be of the form

$$a(\rho_1^2+\rho_2^2+\ldots+\rho_{n-2}^2)+2a_1\rho_1+2a_2\rho_2+\ldots+a_{n-2}\rho_{n-2}+b,$$

where the coefficients satisfy the relation $\Sigma a_i^2=ab$. The coordinates y_i expressed in terms of the ρ's are:

$$y_1=\frac{\rho_1+\frac{a_1}{a}}{h},\ldots,y_{n-2}=\frac{\rho_{n-2}+\frac{a_{n-2}}{a}}{h}. \tag{28}$$

The differential equations for determining the λ's are

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=\frac{2(\lambda_k-\lambda_{k'})(a\rho_{k'}+a_{k'})}{h},$$

the general integral of which is

$$\lambda_k=h\phi_k''-2\sum_1^{n-2}\phi_i'(a\rho_i+a_i)+2a\sum_1^{n-2}\phi_i;$$

integrating the systems (17) and (19) we find that F is of the form $F=\frac{2\Sigma\phi_i}{h}$, a result which might also have been obtained by integrating the system of differential equations

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{2(a\rho_{k'}+a_{k'})}{h}\frac{\partial\theta}{\partial\rho_k}+\frac{2(a\rho_k+a_k)}{h}\frac{\partial\theta}{\partial\rho_{k'}}=0,$$

of which the y's are particular solutions. The tangential equation of the surface is

$$2\left(\rho_1+\frac{a_1}{a}\right)X_1+2\left(\rho_2+\frac{a_2}{a}\right)X_2+\ldots+\left(h-\frac{1}{a}\right)X_{n-1}+2\Sigma\phi_i=0, \tag{24'}$$

from which it appears that no new surfaces are obtained, since (24) may be transformed into (24′) by taking as new ρ's linear functions of the old such that $\rho_i=-(a\rho_i'+a_i)$. It may also be observed that the equations (28) define an inversion in the space S_{n-2}.

§ 4. *Surfaces whose Lines of Curvature are Plane in all Systems.*

The surfaces thus far obtained are not the only ones having their lines of curvature plane in all the $n-2$ systems. The spherical representation of the lines of curvature of such surfaces must consist of an $n-2$-fold orthogonal system of circles on the sphere. The orthogonal system corresponding to the lines of curvature on the surface (23) is a special one, the circles lying in $n-2$

* Darboux, *loc. cit.*, pp. 166–168.

systems of planes which meet at $x_1=x_2=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=1$ and all the circles of any one system being tangent to each other at this point. The $n-2$ lines of intersection of the planes of the $n-2$ systems are parallel to the coordinate axes $x_1, x_2, \ldots\ldots, x_{n-2}$ and perpendicular to the x_{n-1} axis. We may obtain a general system in the following manner: We pass an $n-r$-fold system of $(r-1)$-flats through a point on the x_{n-1}-axis at a distance from the origin$=a$, all the flats of the system having as common axis the flat $x_1=x_2=\ldots\ldots=x_{n-r}=0$, $x_{n-1}=a$. We also pass an $r-2$-fold system of $(n-r+1)$-flats through a point on the x_{n-1}-axis at a distance $\frac{1}{a}$ from the origin, the common axis being the flat $x_{n-r+1}=x_{n-r+2}=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=\frac{1}{a}$. *We shall prove that these two systems of flats will determine on the sphere an $n-2$-fold orthogonal system of circles.* This proposition is an extension to n-space of an analogous one for 3-space. If $a=1$ we get the spherical representation of lines of curvature of the surfaces (24), and for $a=0$ the system is analogous to that of meridians and circles of latitude on a sphere in 3-space.

We express the coordinates x_i of the sphere in terms of the curvilinear coordinates θ_i as follows:

$$\left.\begin{aligned}
x_1&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\sin\theta_1}{1+a\sin\theta_{n-2}\cos_{n-r}},\\
x_2&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\cos\theta_1}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r}&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
x_{n-r+1}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-3}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\quad x_{n-1}=\frac{a+\cos\theta_{n-r}\sin\theta_{n-2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},
\end{aligned}\right\}\quad(29)$$

from which it is easily verified that $\Sigma x_i^2=1$. In order to prove that the system is orthogonal we need only calculate the linear element $d\sigma$. A rather long, but not inherently difficult calculation will show that this element is

$$d\sigma^2=E_1d\theta_1^2+E_2d\theta_2^2+\ldots\ldots+E_{n-r}d\theta_{n-r}^2+E_{n-r+1}d\theta_{n-r+1}^2 \\ +\ldots\ldots+E_{n-3}d\theta_{n-3}^2+E_{n-2}d\theta_{n-2}^2, \quad (30)$$

where the E's have the following values:

$$\left.\begin{aligned}
E_1 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_2}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_2 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_3}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-r} &= \frac{(1-a^2)\sin^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}, \quad E_{n-r+1}=\frac{(1-a^2)\cos^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_{n-r+2} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-3} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}\ldots\cos^2\theta_{n-4}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2},\\
E_{n-2} &= \frac{(1-a^2)}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}.
\end{aligned}\right\} \quad (31)$$

That the curves θ_i are circles appears from the following systems of equations deduced from (29):

$$\left.\begin{aligned}
x_1 &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{\cos\theta_{n-r}}(x_{n-1}-a),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r} &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}}{\cos\theta_{n-r}}(x_{n-1}-a),\\
x_{n-r+1} &= -\frac{a\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2} &= -\frac{a\cos\theta_{n-2}\sin\theta_{n-r+1}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),
\end{aligned}\right\} \quad (32)$$

and that the geometrical construction given at the beginning of this section is correct appears at once from these equations.

Since the coordinates y_i in terms of the x's are given by the formulae:

$$y_1=\frac{x_1}{1+x_{n-1}},\quad y_{n-2}=\frac{x_{n-2}}{1+x_{n-1}},\quad x_1=\frac{2y_1}{1+\Sigma y_i^2},\ \ldots,\ x_{n-1}=\frac{1-\Sigma y_i^2}{1+\Sigma y_i^2}, \quad (33)$$

we have

$$\left.\begin{aligned}
y_1 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_2 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\cos\theta_1}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-r} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_{n-r+1} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-2} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
1-\Sigma y^2 &= \frac{2(a+\cos\theta_{n-r}\sin\theta_{n-2})}{(1+a)(1+\cos\theta_{n-r}\sin\theta_{n-2})},\\
\frac{2}{1+\Sigma y^2} &= \frac{(a+1)(1+\cos\theta_{n-r}\sin\theta_{n-2})}{1+a\cos\theta_{n-r}\sin\theta_{n-2}}.
\end{aligned}\right\} \tag{34}$$

The coordinates y_i satisfy the following system of $\frac{n-2\cdot n-3}{2}$ differential equations:

$$\left.\begin{aligned}
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\cot\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad (k,\,k'=1,\,2,\,\ldots,\,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\tan\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad (k,\,k'=n-r+1,\,\ldots,\,n-r),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}-\frac{\cot\theta_{n-r}+\csc\theta_{n-r}\sin\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad (k=1,\,2,\,\ldots,\,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}=0,\quad \begin{Bmatrix}k=1,\,2,\,\ldots,\,n-r-1\\ k'=n-r+1,\,\ldots,\,n-3\end{Bmatrix},\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}+\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad (k=n-r+1,\,\ldots,\,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad (k=1,\,2,\,\ldots,\,n-r+1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}+\frac{\tan\theta_{n-2}+\cos\theta_{n-r}\sec\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta}=0,\quad (k=n-r+1,\,\ldots,\,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_{n-r}\partial\theta_{n-2}}-\frac{\sin\theta_{n-2}\sin\theta_{n-r}}{1+\cos_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-r}}=0.\quad k\neq k'.
\end{aligned}\right\} \tag{35}$$

The general solution of this system is

$$F=\frac{\Sigma\Phi_i\sqrt{E_i}(1+\Sigma y_i^2)}{1+\cos\theta_{n-r}\sin\theta_{n-2}}, \tag{36}$$

where Φ_i is a function of θ_i alone and the *E's* have the values given in (31). The surface is therefore the envelope of the flats

$$2y_1X_1+2y_2X_2+\ldots\ldots+2y_{n-r}X_{n-r}+2y_{n-r+1}X_{n-r+1}+\ldots\ldots \\ +(1-\Sigma y_i^2)X_{n-1}=(1+\Sigma y_i^2)\Sigma\Phi_i\sqrt{E_i}. \quad (37)$$

For any constant value of r all the surfaces obtained by putting for Φ_i arbitrary functions of θ_i have the same spherical representation of their lines of curvature. Since r can have any value from 3 to $n-1$ *there will be* $n-3$ *different types.* To these types must also be added the type (24) corresponding to $a=1$ obtained in § 3, viz.:

$$2\rho_1X_1+2\rho_2X_2+\ldots\ldots+2\rho_{n-r}X_{n-r}+\ldots\ldots+(1-\Sigma\rho_i^2)X_{n-1}+2\Sigma\phi_i+2c=0. \quad (24)$$

The surfaces (37) may be considered as the envelope of the radical flats of the two spheres:

$$\left.\begin{aligned}
&\sum_1^{n-1}X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_{n-r+1}+\ldots\ldots \\
&\quad+2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2}+\frac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1}=2\Phi_{n-2}\csc\theta_{n-2} \\
&\quad+2\Phi_{n-3}\cot\theta_{n-2}\ldots\cos\theta_{n-4}+\ldots\ldots+2\Phi_{n-r+2}\cot\theta_{n-2}\cos\theta_{n-r+1} \\
&\quad+2\Phi_{n-r+1}\cot\theta_{n-2}+C, \\
&\sum_1^{n-1}X_i^2-2\sin\theta_{n-r}\ldots\sin\theta_1X_1-\ldots\ldots-2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r} \\
&\quad-\frac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1}=-2\Phi_{n-r}-2\Phi_{n-r-1}\sin\theta_{n-r}-\ldots\ldots \\
&\quad-2\Phi_1\sin\theta_{n-r}\ldots\sin\theta_2+C,
\end{aligned}\right\}\quad(38)$$

the centres of which lie respectively on the hyperboloid of revolution

$$X_{n-r+1}^2+X_{n-r+2}^2+\ldots\ldots+X_{n-2}^2-\frac{X_{n-1}^2}{\frac{a^2}{1-a^2}}=-1,\quad X_i=0,\quad i=1,2,\ldots\ldots,n-r, \quad (39)$$

and on the ellipsoid

$$\sum_1^{n-r}X_i^2+\frac{X_{n-1}^2}{\frac{1}{1-a^2}}=1,\qquad X_{n-r+1}=0,\ \ldots\ldots,X_{n-2}=0. \quad (40)$$

These quadrics are focal, the vertex of one passing through the focus of the other. They are, moreover, perpendicular to each other, having the X_{n-1}-axis in common.*

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* The two spaces in which the quadrics are immersed exhibit the maximum of perpendicularity expressed by the fraction $\frac{r-2}{r-1}$; See Schoute, *Mehrdimensionale Geometrie*, 1. Theil, p. 49.

The surfaces (24) may also be considered as the envelope of the two spheres

$$\left.\begin{aligned}
&\text{a)}\ \Sigma X_i^2+4\rho_1 X_1+\ldots\ldots+4\rho_{n-r}X_{n-r}+(1-2\sum_1^{n-r}\rho_i^2)X_{n-1}\\
&\qquad\qquad=-4\phi_1-\ldots\ldots-4\phi_{n-r}+C-2c,\\
&\text{b)}\ \Sigma X_i^2-4\rho_{n-r+1}X_{n-r+1}-\ldots\ldots-4\rho_{n-2}X_{n-2}-(1-2\sum_{n-r+1}^{n-2}\rho_i^2)X_{n-1}\\
&\qquad\qquad=4\phi_{n-r+1}+\ldots\ldots+4\phi_{n-2}+C+2c,
\end{aligned}\right\}\quad(41)$$

the centres of which lie on the paraboloids

$$\left.\begin{aligned}
&2X_{n-1}=2-\frac{\sum_1^{n-r}X_i^2}{4},\quad X_i=0;\quad (i=n-r+1,\ldots\ldots,n-2),\\
&2X_{n-1}=2-\frac{\sum_{n-r+1}^{n-2}X_i^2}{4},\quad X_i=0;\quad (i=1,2,\ldots\ldots,n-r).
\end{aligned}\right\}\quad(42)$$

§ 5. *The Generalized Dupin Cyclides in* S_{n-1}.

If in (37) we put $\Phi_1=\Phi_2=\Phi_3=\ldots\ldots=\Phi_{n-r-1}=0,\ \Phi_{n-r}=1-\frac{ka}{\sqrt{1-a^2}}\cos\theta_{n-r}$, $\Phi_{n-r+1}=\Phi_{n-r+2}=\ldots\ldots=\Phi_{n-3}=0,\ \Phi_{n-2}=-\frac{k}{\sqrt{1-a^2}}$, and then derive the envelope of the resulting flat in the usual way we get the surface in the following parametric form:

$$\left.\begin{aligned}
X_1&=\frac{\sin\theta_{n-r}\ldots\ldots\sin\theta_2\sin\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_2&=\frac{\sin\theta_{n-r}\ldots\ldots\sin\theta_2\cos\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-r}&=\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_{n-r+1}&=\frac{-\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-4}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-2}&=-\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
X_{n-1}&=\frac{\sqrt{1-a^2}}{a}\cdot\frac{a\cos\theta_{n-r}+k\sqrt{1-a^2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}}-\frac{k}{a}.
\end{aligned}\right\}\quad(43)$$

Eliminating the parameters θ_i we obtain the quartic surface

$$(1-a^2)\,[\sum_1^{n-1}X_i^2-k^2-1]^2=4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)\,(\sum_{n-r+1}^{n-2}X_i^2)\right],\quad(44)$$

which, as we shall prove, is a generalization of the Dupin cyclide of ordinary space. The focal sheets of the surface are given by the formulae

$$\overline{X}_i = X_i + R_k x_i,$$

where R_k are the $n-2$ principal radii of curvature calculated from the equations of Olinde Rodrigue:

$$\frac{\partial X_i}{\partial \theta_k} + R_k \frac{\partial x}{\partial \theta_k}, \qquad \begin{Bmatrix} i=1, 2, \ldots, n-1 \\ k=1, 2, \ldots, n-2 \end{Bmatrix},$$

where x_i have the values given in (29). Although a general surface (37) will have $n-2$ different focal sheets, in the case of the surface (43) it is found that *only two focal sheets exist*, viz.,

$$\left.\begin{array}{l} \text{a) } X_1^2+X_2^2+\ldots\ldots+X_{n-r}^2+\dfrac{X_{n-1}^2}{\dfrac{1}{1-a^2}}=1,\ X_i=0,\quad i=n-r+1,\ldots,n-2, \\ \text{b) } X_{n-r+1}^2+X_{n-r+2}^2+\ldots\ldots+X_{n-2}^2-\dfrac{X_{n-1}^2}{\dfrac{a}{1-a^2}}=-1,\ X_i=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1, 2, \ldots, n-r, \end{array}\right\} \tag{45}$$

a pair of focal quadrics, of which one is an ellipsoid immersed in the space $X_{n-r+1}=X_{n-r+2}=\ldots\ldots=X_{n-2}=0$, and the other a hyperboloid immersed in the space $X_1=X_2=\ldots\ldots=X_{n-r}=0$. If in (38) we put $\Phi_1=\Phi_2=\ldots\ldots=\Phi_{n-r-1}=0$, $\Phi_{n-r}=1-\dfrac{ak}{\sqrt{1-a^2}}\cos\theta_{n-r}$, $\Phi_{n-r+1}=\ldots\ldots=\Phi_{n-3}=0$, $\Phi_{n-2}=\dfrac{-k}{\sqrt{1-a^2}}$, and $C=1+k^2$, we have the two spheres,

$$\left.\begin{array}{l} \text{a) } \sum_1 X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-3}X_{n-r+1}+\ldots\ldots \\ \qquad +2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2}+\dfrac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1}=1+k^2-\dfrac{2k}{\sqrt{1-a^2}}\csc\theta_{n-2}, \\ \text{b) } \sum X_i^2-2\sin\theta_{n-r}\ldots\ldots\sin\theta_1 X_1-\ldots\ldots-2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r} \\ \qquad -\dfrac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1}=1+k^2-2\left[1-\dfrac{ak}{\sqrt{1-a^2}}\cos\theta_{n-r}\right], \end{array}\right\} \tag{46}$$

the radii R_1 and R_2 of which are

$$R_1=\pm\left[\frac{\csc\theta_{n-2}}{\sqrt{1-a^2}}-k\right],\quad R_2=\pm\left[\frac{a}{\sqrt{1-a^2}}\cos\theta_{n-r}+k\right],$$

from which we derive

$$R_1+R_2=\pm\left[\frac{\cos\theta_{n-2}+a\cos\theta_{n-r}}{\sqrt{1-a^2}}\right]=D. \tag{47}$$

Hence, *the surface* (43) *is the envelope of two sets of spheres that touch, and whose centres lie respectively on the ellipsoid* (45a) *and hyperboloid* (45b). *The surface is thus a generalization of the Dupin cyclide in ordinary space.* If k varies we get parallel cyclides as appears from (47).

If we give to r in (43) the successive values $3, 4, \ldots, n-1$ we obtain $n-3$ different types of cyclides.* Thus, if $n=3$ the focal hyperboloid becomes a hyperbola lying in the plane $X_1=X_2=\ldots=X_{n-3}=0$, while the ellipsoid is $n-3$ dimensional and is immersed in the space $X_{n-2}=0$. If S_{n-1} is an odd space, and $r=\frac{n+2}{2}$, the focal quadrics are of the same dimension $=\frac{n-2}{2}$, and the equation of the corresponding cyclide is

$$(1-a^2)\,[\Sigma X_i^2-k^2-1]^2$$
$$=4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)\,(X^2_{\frac{n}{2}}+X^2_{\frac{n+2}{2}}+\ldots+X^2_{n-2})\right]. \quad (48)$$

§ 6. We shall next discuss the type of cyclides for which $a=1$. In (24) we put $\phi_i=\frac{k+2}{4}\rho_i^2$, $i=1, 2, \ldots, n-r$, $\phi_{n-r+k}=\frac{k-2}{4}\rho^2_{n-r+k}$, $k=1, 2, \ldots, r-2$, $c=\frac{k}{4}$, and obtain the surface in the following parametric form:

$$\left.\begin{aligned}
&2X_i-2\rho_iX_{n-1}+(k+2)\rho_i=0, \qquad i=1, 2, \ldots, n-r,\\
&2X_{n-r+k}-2\rho_{n-r+k}X_{n-1}+(k-2)\rho_{n-r+k}=0, \qquad k=1, 2, \ldots, r-2,\\
&X_{n-1}=\frac{\frac{k+2}{2}\sum\limits_1^{n-r}\rho_i^2+\frac{k-2}{2}\sum\limits_{n-r+1}^{n-2}\rho_i^2-\frac{k}{2}}{1+\Sigma\rho_i^2},
\end{aligned}\right\} \quad (49)$$

or, in Cartesian form,

$$2X_{n-1}\left(\sum_1^{n-1}X_i^2+\frac{k^2-4}{4}\right)-(k-2)\sum_1^{n-r}X_i^2$$
$$-(k+2)\sum_{n-r+1}^{n-2}X_i^2-kX^2_{n-1}+\frac{k(k^2-4)}{4}=0. \quad (49')$$

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* The types for which $r>\frac{n+2}{2}$ do not differ essentially from those for which $r<\frac{n+2}{2}$; the focal ellipsoid and hyperboloid merely interchange. We need therefore consider only the surfaces of type $r\leqq\frac{n+2}{2}$.

The two sets of spheres (41) are, putting $C=\frac{k^2}{4}$,

$$\left.\begin{aligned}\sum_1^{n-1} X_i^2+4\rho_1X_1+\ldots+4\rho_{n-r}X_{n-r}+(1-2\sum_1^{n-r}\rho_i^2)X_{n-1}&\\ =-(k+2)\sum_1^{n-r}\rho_i^2+\frac{k^2}{4}-\frac{k}{2}&,\\ \sum_1^{n-1} X_i^2-4\rho_{n-r+1}X_{n-r+1}-\ldots-4\rho_{n-3}X_{n-3}-(1-2\sum_{n-r+1}^{n-2}\rho_i^2)X_{n-1}&\\ =(k-2)\sum_{n-r+1}^{n-2}\rho_i^2+\frac{k^2}{4}+\frac{k}{2}&,\end{aligned}\right\}\quad(50)$$

and the radii are, respectively:

$$R_1=\pm\tfrac{1}{2}(1+2\sum_1^{n-r}\rho_i^2-k),\quad R_2=\pm\tfrac{1}{2}(1+2\sum_{n-r+1}^{n-2}\rho_i^2+k). \qquad (51)$$

$\therefore R_1+R_2=\pm(1+\sum_1^{n-2}\rho_i^2)=$ Distance between the centres.

Hence, *the cyclide of the third order is the envelope of two sets of spheres that touch, and whose centres lie respectively on two focal paraboloids of revolution whose equations are*

$$\left.\begin{aligned}4X_{n-1}+2&=\sum_{n-r+1}^{n-2}X_i^2,\quad X_1=X_2=X_{n-r}=0,\\ 4X_{n-1}-2&=-\sum_1^{n-r}X_i^2,\quad X_{n-r+1}=X_{n-r+2}=\ldots=X_{n-2}=0.\end{aligned}\right\}\quad(52)$$

If $r=3$, one paraboloid is one-dimensional, *i. e.*, a parabola, while the other is of dimension $n-3$. If S_{n-1} is an odd space and $r=\frac{n+2}{2}$, both paraboloids are of the same dimension, viz.: $\frac{n}{2}$. All the cyclides obtained by putting $r=3,\ldots,n=1$ in succession in (49) have the same spherical representation, while in the case of the hyperbolic-elliptic types we get a different orthogonal system of circles on the Gaussian sphere for each value of r.* If k varies we have a system of parallel cyclides as before.

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*Schoute in his *Mehrdimensionale Geometrie*, II. Theil, pp. 316–320, has by a synthetic method borrowed from 3-space, derived the type for which $r=n-1$, *i. e.*, where the focal quadrics are an ellipse and a hyperboloid of revolution of $n-3$ dimensions. That this method leads to the n-dimensional generalization of Dupin's cyclide in the sense that we get *all* such cyclides is, however, not true, although the opposite might be inferred from the author's statement. On p. 320 the author mentions the existence of two families of spheres of the general nature considered above, but does not consider their envelope. The author's reference to a short article in *Verslagen der Akademie von Amsterdam*, vom Februar 1905, I have not been able to look up.

§ 7. *Some Geometrical Properties of the Cyclides in General Space.*

The loci on the cyclide where the radii of the two sets of spheres equal zero will be spreads of double points on the surface. We obtain these point-spheres by equating R_1 and R_2 to zero giving

$$\csc\theta_{n-2}=k\sqrt{1-a^2},\quad \cos\theta_{n-r}=-\frac{k}{a}\sqrt{1-a^2}.$$

Substituting the values of $\cot\theta_{n-2}$ and $\csc\theta_{n-2}$ in (43) and eliminating the remaining parameters θ_i, we get an r—3-dimensional sphere whose equations are

$$X_1=X_2=\ldots\ldots=X_{n-r}=0,\quad \sum_{n-r+1}^{n-2}X_i^2=k^2(1-a^2)-1,\quad X_{n-1}=ka, \tag{53}$$

all the points of which must be considered as centres of null-spheres. Again, substituting the values of $\cos\theta_{n-r}$ and $\sin\theta_{n-r}$ in (43) we get a second locus of double points, viz.:

$$\sum_1^{n-r}X_i^2=\frac{a^2-k^2(1-a^2)}{a^2},\quad X_{n-r+1}=X_{n-r+2}=\ldots\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{k}{a}, \tag{54}$$

(53) lies on the focal hyperboloid H_{r-2} and (54) on the focal ellipsoid E_{n-r}. We shall denote these spheres by Σ_{r-3} and Σ_{n-r-1}, and the two systems of spheres by S_h and S_e, the spheres S_h having their centres on H_{r-2}, and the spheres S_e on E_{n-r}. Since all the point-spheres having their centres on Σ_{r-3} are tangents to all the spheres S_h, their centres must belong to these spheres, hence all the spheres S_h pass through the locus Σ_{r-3}; similarly all spheres S_e pass through the locus Σ_{n-r+1}. We may therefore say that *the quartic cyclides in S_{n-1} is the envelope of spheres having their centres on the ellipsoid E_{n-r} and passing through the sphere-locus Σ_{r-3} on H_{r-2}*; or, *it is the envelope of spheres having their centres on the hyperboloid H_{r-2} and passing through the sphere-locus Σ_{n-r-1} on the ellipsoid E_{n-r}*.

Consider the flat

$$u_1X_1+u_2X_2+\ldots\ldots+u_{n-1}X_{n-1}+p=0; \tag{55}$$

the condition that it shall be tangent to H_{r-2} and E_{n-r} are

$$\sum_1^{n-r}u_i^2+\frac{1}{1-a^2}u_{n-1}^2=p^2,\quad -\sum_{n-r+1}^{n-2}u_i^2+\frac{a^2}{1-a^2}u_{n-1}^2=p^2, \tag{56}$$

from which it follows that we must have $\sum_1^{n-1}u_i^2=0$, which means that *the flat is isotropic*. Introducing in (55) the parameters

$$u_1=2\rho_1,\ \ldots\ldots,\ u_{n-3}=2\rho_{n-3},\quad u_{n-2}=1-\Sigma\rho_i^2,\quad u_{n-1}=i(1+\Sigma\rho_i^2),$$

we obtain the ∞^{n-3} flats

$$2\rho_1 X_1 + 2\rho_2 X_2 + \ldots + (1-\Sigma\rho_i^2) X_{n-2} + i(1+\Sigma\rho_i^2) X_{n-1} + p = 0. \tag{57}$$

where p has the value

$$p = \pm\sqrt{4\sum_1^{n-r}\rho_i^2 - \frac{(1+\Sigma\rho^2)^2}{1-a^2}},$$

The equations of the "developable" surface obtained by letting the ρ's vary are as follows:

$$\left.\begin{aligned} X_i &= \frac{-2i\rho_i}{1+\Sigma\rho^2} X_{n-1} + \frac{2a^2\rho_i}{p(1-a^2)}, \qquad (i=1, 2, \ldots, n-r), \\ X_i &= \frac{-2i\rho_i}{1+\Sigma\rho_i^2} X_{n-1} + \frac{2\rho_i}{p(1-a^2)}, \qquad (i=n-r+1, \ldots, n-3), \\ X_{n-2} &= -\frac{i(1-\Sigma\rho_i^2)}{1+\Sigma\rho_i^2} X_{n-1} + \frac{1-\Sigma\rho_i^2}{p(1-a^2)}. \end{aligned}\right\} \tag{58}$$

In a former paper* we have studied isotropic complexes of which (58) is a special case. Since the complex contains ∞^{n-3} isotropic lines we shall denote it by the symbol Δ_{n-3}. By a theorem proved in A. I., p. 207, we know that it has a focal surface which is the envelope of the ∞^{n-4} isotropic 2-spreads contained in it, and that this focal surface has in general $n-3$ sheets. The edges of regression on the surface are minimal curves satisfying a certain system of differential equations. In the special case of (58) it may be proved that *there are only three sheets;* in fact, the determinant equation (13), p. 205, l. c., which determines the number of sheets, reduces to the simple form

$$(\sigma-A)^{n-r-1}(\sigma-B)^{r-3}\left[\sigma - A - \frac{16a^2\sum_1^{n-r}\rho_i^2}{p^3(1-a^2)}\right] = 0, \tag{59}$$

where

$$A = \frac{4}{p} - \frac{2(1+\Sigma\rho_i^2)}{p(1-a^2)}, \quad B = \frac{-2(1+\Sigma\rho_i^2)}{p(1-a^2)}.$$

Substituting the values of σ in equations (60), footnote, we obtain the three focal sheets, *two of which are seen to be precisely the focal quadrics* H_{r-3} and

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*A, §§ 1–5 where the formulae developed are true for any space, even or odd, if instead of the parameters α_i and β_i we use ρ_i (see II, § 1, of this paper). The equation determining the focal sheets becomes the equation (26), note, when written of order $n-3$. The equations of the focal sheets take the form:

$$\left.\begin{aligned} X_i &= \frac{\rho_i\sigma_k}{2} - \frac{p}{2}, \quad X_{n-1} = \frac{1-\Sigma\rho_i^2}{4}\sigma_k + \frac{1}{2}\Sigma p_i\rho_i - \frac{p}{2}, \\ X_{n-1} &= i\left[\frac{1+\Sigma\rho_i^2}{4}\sigma_k - \frac{1}{2}\Sigma p_i\rho_i + \frac{p}{2}\right], \quad \begin{Bmatrix} i=1, 2, \ldots, n-3 \\ k=1, 2, 3 \end{Bmatrix}. \end{aligned}\right\} \tag{60}$$

E_{n-r}. The surface of reference of Δ_{n-3} is the imaginary ellipsoid in the space $X_{n-1}=0$:

$$\frac{(1-a^2)}{a^2}\sum_1^{n-r} X_i^2+(1-a^2)\sum_{n-r-1}^{n-2} X_i^2+1=0, \quad X_{n-1}=0 \tag{61}$$

obtained by putting $X_{n-1}=0$ in (58). We have thus proved the

THEOREM. *The tangent $n-2$-flats common to the two focal quadrics H_{r-3} and E_{n-r} generate an isotropic complex Δ_{n-3} which has H_{n-3} and E_{n-r} for two of its three focal sheets and the imaginary quadric* (61) *for surface of reference.*

We shall now return to the study of the quartic cyclide (44). Any straight line $\overline{BA}$ joining a point A on Σ_{r-4} to a point B on Σ_{n-r-1} is an isotropic line. In fact, from (53) and (54) we find that the distance is:

$$D=\sqrt{k^2(1-a^2)-1+\frac{a^2-k^2(1-a^2)}{a^2}+k^2\left(a-\frac{1}{a}\right)^2}=0.$$

The ∞^{n-4} isotropic lines generate a locus D_{n-4} which is the intersection of the quadric cylinders

$$\sum_1^{n-r} X_i^2=\frac{a^2r_2^2}{k^2(1-a^2)^2}[X_{n-1}+ak]^2,\ \sum_{n-r+1}^{n-2} X_i^2=\frac{a^2r_1^2}{k^2(1-a^2)}\left[X_{n-1}+\frac{k}{a}\right]=0.^* \tag{62}$$

If we substitute the values of $\sum_1^{n-r} X_i^2$ and $\sum_{n-r+1}^{n-2} X_i^2$ from these equations in (44) we find that it is identically satisfied, hence *the ∞^{n-4} isotropic lines $\overline{AB}$ lie on the cyclide.* But they also belong to the isotropic complex Δ_{n-3}, hence

THEOREM. *The quartic cyclides in S_{n-1} is inscribed in the isotropic complex Δ_{n-3}, the locus of contact being the intersection of two quadric cylinders* (62).

Since the lines $\overline{AB}$ are lines of contact between the point-spheres on Σ_{r-4} and Σ_{n-r-1} and the cyclides, they are lines of curvature.

Consider any two points A and B on Σ_{r-3} and Σ_{n-r-1}. The two tangent flats to H_{r-2} and E_{n-r} at A and B respectively intersect the X_{n-1}-axis in a point C whose coordinates are:

$$X_1=X_2=\ldots\ldots=X_{n-2}=0, \quad X_{n-1}=\frac{-a}{k(1-a^2)}.$$

The sphere having its centre at this point and radius equal to $\overline{CB}$ (or $\overline{CA}$) is

$$\Sigma X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{[1-k^2(1-a^2)][a^2-k^2(1-a^2)]}{k^2(1-a^2)^2}, \tag{63}$$

* r_1 and r_2 in these equations are the radii of the spheres (53) and (54).

which, as is easily seen from equations (53) and (54), contains the loci of double points Σ_{r-3} and Σ_{n-r-1}. But the isotropic line $\overline{BA}$ is normal to the radius $\overline{CB}$ (or $\overline{CA}$), and, since it is isotropic, it must lie on the sphere. Hence, the sphere (64) is tangent to the cyclide along the ∞^{n-4} isotropic lines $\overline{AB}$. We shall call this sphere *the principal sphere.**

The point-spheres on Σ_{r-3} intersect the space in which is immersed the focal hyperboloid H_{r-2} in an $r-2$-dimensional sphere K_{r-2} whose equations are:

$$\sum_{n-r+1}^{n-2} X_i^2 + \left(X_{n-1} + \frac{k}{a}\right)^2 = \frac{k^2(1-a^2)-a^2}{a^2}, \quad X_i=0, \quad i=1, 2, \ldots, n-r, \quad (64)$$

which is tangent to H_{r-2} along Σ_{r-3}. In the same way, the point-spheres having their centres on Σ_{n-r-1}, intersect the space in which is immersed the focal ellipsoid E_{n-r} in a sphere K_{n-r} of $n-r$ dimensions whose equations are:

$$\sum_{1}^{n-r} X_i^2 + (X_{n-1}+ka)^2 = 1-k^2(1-a^2), \quad X_i=0, \quad i=n-r+1, \ldots, n-2, \quad (65)$$

and this sphere is tangent to the focal ellipsoid E_{n-r} along Σ_{n-r-1}. The proof being very simple need not be given here. We shall call the spheres K_{r-2} and K_{n-r} *the focal loci of the cyclide.** *The focal loci are normal to the principal sphere* as may easily be proved from equations (63), (64) and (65).

We have seen that all the spheres S_e having their centres on the focal ellipsoid E_{n-r} pass through Σ_{r-3} and that all the spheres S_h having their centres on H_{r-2} pass through Σ_{n-r-1}. Consider now a cone with vertex at any point A on Σ_{r-3} and passing through the ellipsoid E_{n-r}. This cone is a cone of revolution whose axis is a tangent to the hyperboloid at A, which, as we have seen, passes through the centre C of the principal sphere. In fact, we prove that the radii of the spheres S_e which meet at A, make a constant angle with this tangent. The direction-cosines of any line PA, where P is any point on E_{n-r}, are:

$$\frac{X_1}{PA}, \ \frac{X_2}{PA}, \ldots, \frac{X_{n-r}}{PA}, \ \frac{-\overline{X}_{n-r+1}}{PA} \ldots \frac{-\overline{X}_{n-2}}{PA}, \ \frac{X_{n-1}-\overline{X}_{n-1}}{PA},$$

where $X_1, X_2, \ldots, X_{n-r}X_{n-1}$ are the coordinates of the point P on the ellipsoid, and $\overline{X}_{n-r+1}, \ldots, \overline{X}_{n-2}, \overline{X}_{n-1}$ those of a point on Σ_{n-3}. Again, the direction-cosines of the tangent to the hyperboloid at A are:

$$0, 0, \ldots, 0, \ \frac{\overline{X}_{n-r+1}}{CA}, \ldots, \frac{\overline{X}_{n-2}}{CA}, \ \frac{-a\left[k-\dfrac{1}{k(1-a^2)}\right]}{CA}.$$

*In adopting the above nomenclature we have followed the analogous one used by Darboux, *Leçons sur les systèmes orthogonaux*, p. 489.

E_{n-r}. The surface of reference of Δ_{n-3} is the imaginary ellipsoid in the space $X_{n-1}=0$:

$$\frac{(1-a^2)}{a^2}\sum_{1}^{n-r} X_i^2+(1-a^2)\sum_{n-r-1}^{n-2} X_i^2+1=0,\quad X_{n-1}=0 \tag{61}$$

obtained by putting $X_{n-1}=0$ in (58). We have thus proved the

THEOREM. *The tangent $n-2$-flats common to the two focal quadrics H_{r-3} and E_{n-r} generate an isotropic complex Δ_{n-3} which has H_{n-3} and E_{n-r} for two of its three focal sheets and the imaginary quadric* (61) *for surface of reference.*

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$$D=\sqrt{k^2(1-a^2)-1+\frac{a^2-k^2(1-a^2)}{a^2}+k^2\left(a-\frac{1}{a}\right)^2}=0.$$

The ∞^{n-4} isotropic lines generate a locus D_{n-4} which is the intersection of the quadric cylinders

$$\sum_{1}^{n-r} X_i^2=\frac{a^2r_2^2}{k^2(1-a^2)^2}[X_{n-1}+ak]^2,\quad \sum_{n-r+1}^{n-2} X_i^2=\frac{a^2r_1^2}{k^2(1-a^2)}\left[X_{n-1}+\frac{k}{a}\right]=0.^* \tag{62}$$

If we substitute the values of $\sum_{1}^{n-r} X_i^2$ and $\sum_{n-r+1}^{n-2} X_i^2$ from these equations in (44) we find that it is identically satisfied, hence *the ∞^{n-4} isotropic lines $\overline{AB}$ lie on the cyclide.* But they also belong to the isotropic complex Δ_{n-3}, hence

THEOREM. *The quartic cyclides in S_{n-1} is inscribed in the isotropic complex Δ_{n-3}, the locus of contact being the intersection of two quadric cylinders* (62).

Since the lines $\overline{AB}$ are lines of contact between the point-spheres on Σ_{r-4} and Σ_{n-r-1} and the cyclides, they are lines of curvature.

Consider any two points A and B on Σ_{r-3} and Σ_{n-r-1}. The two tangent flats to H_{r-2} and E_{n-r} at A and B respectively intersect the X_{n-1}-axis in a point C whose coordinates are:

$$X_1=X_2=\ldots\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{-a}{k(1-a^2)}.$$

The sphere having its centre at this point and radius equal to $\overline{CB}$ (or $\overline{CA}$) is

$$\Sigma X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{[1-k^2(1-a^2)]\,[a^2-k^2(1-a^2)]}{k^2(1-a^2)^2}, \tag{63}$$

* r_1 and r_2 in these equations are the radii of the spheres (53) and (54).

which, as is easily seen from equations (53) and (54), contains the loci of double points Σ_{r-3} and Σ_{n-r-1}. But the isotropic line $\overline{BA}$ is normal to the radius $\overline{CB}$ (or $\overline{CA}$), and, since it is isotropic, it must lie on the sphere. Hence, the sphere (64) is tangent to the cyclide along the ∞^{n-4} isotropic lines $\overline{AB}$. We shall call this sphere *the principal sphere.**

The point-spheres on Σ_{r-3} intersect the space in which is immersed the focal hyperboloid H_{r-2} in an $r-2$-dimensional sphere K_{r-2} whose equations are:

$$\sum_{n-r+1}^{n-2} X_i^2+\left(X_{n-1}+\frac{k}{a}\right)^2=\frac{k^2(1-a^2)-a^2}{a^2},\quad X_i=0,\quad i=1,2,\ldots,n-r, \quad (64)$$

which is tangent to H_{r-2} along Σ_{r-3}. In the same way, the point-spheres having their centres on Σ_{n-r-1}, intersect the space in which is immersed the focal ellipsoid E_{n-r} in a sphere K_{n-r} of $n-r$ dimensions whose equations are:

$$\sum_{1}^{n-r} X_i^2+(X_{n-1}+ka)^2=1-k^2(1-a^2),\quad X_i=0,\quad i=n-r+1,\ldots,n-2, \quad (65)$$

and this sphere is tangent to the focal ellipsoid E_{n-r} along Σ_{n-r-1}. The proof being very simple need not be given here. We shall call the spheres K_{r-2} and K_{n-r} *the focal loci of the cyclide.** *The focal loci are normal to the principal sphere* as may easily be proved from equations (63), (64) and (65).

We have seen that all the spheres S_e having their centres on the focal ellipsoid E_{n-r} pass through Σ_{r-3} and that all the spheres S_h having their centres on H_{r-2} pass through Σ_{n-r-1}. Consider now a cone with vertex at any point A on Σ_{r-3} and passing through the ellipsoid E_{n-r}. This cone is a cone of revolution whose axis is a tangent to the hyperboloid at A, which, as we have seen, passes through the centre C of the principal sphere. In fact, we prove that the radii of the spheres S_e which meet at A, make a constant angle with this tangent. The direction-cosines of any line PA, where P is any point on E_{n-r}, are:

$$\frac{X_1}{PA},\ \frac{X_2}{PA},\ldots,\frac{X_{n-r}}{PA},\ \frac{-\bar{X}_{n-r+1}}{PA}\cdots\frac{-\bar{X}_{n-2}}{PA},\ \frac{X_{n-1}-\bar{X}_{n-1}}{PA},$$

where $X_1, X_2, \ldots, X_{n-r}X_{n-1}$ are the coordinates of the point P on the ellipsoid, and $\bar{X}_{n-r+1},\ldots,\bar{X}_{n-2},\bar{X}_{n-1}$ those of a point on Σ_{n-3}. Again, the direction-cosines of the tangent to the hyperboloid at A are:

$$0,0,\ldots,0,\ \frac{\bar{X}_{n-r+1}}{CA},\ldots,\ \frac{\bar{X}_{n-2}}{CA},\ \frac{-a\left[k-\dfrac{1}{k(1-a^2)}\right]}{CA}.$$

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*In adopting the above nomenclature we have followed the analogous one used by Darboux, *Leçons sur les systémes orthogonaux*, p. 489.

The distances $\overline{PA}$ and $\overline{CA}$ are:

$$PA=\frac{a\cos\theta_{n-r}}{\sqrt{1-a^2}}+k,\quad CA=\sqrt{\frac{[1-k^2(1-a^2)][a^2-k(1-a^2)]}{k(1-a^2)}},$$

and the cosine of the angle between these two lines is found to be

$$\cos\omega=\frac{\sqrt{k^2(1-a^2)-1}}{\sqrt{k^2(1-a^2)-a^2}}=\text{const.}$$

In the same way we may prove that the cosine of the angle between the lines $\overline{P'B}$ and $\overline{CB}$ where P' is any point on H_{r-2} and B a point on Σ_{n-r-1}, is constant and equal to

$$\cos\omega'=\frac{\sqrt{a^2-k^2(1-a^2)}}{\sqrt{1-k^2(1-a^2)}}=\sec\omega=\text{const.}$$

and, since $\omega'=\frac{\pi}{2}-\alpha'$, where α' is the angle between the spheres S_h having its centre at P' and the focal locus K_{n-r}, we have

$$\sin\omega'=\cos\alpha'=\frac{i\sqrt{1-a^2}}{\sqrt{k^2(1-a^2)-1}}.$$

Hence, if ω is a real angle, ω' is imaginary and vice versa. We may now state the result in the following

THEOREM. *A general cyclide of Dupin of the fourth order in S_{n-1} is the envelope of spheres whose centres lie on an ellipsoid of revolution E_{n-r} and which intersect a fixed $r-2$-dimensional sphere K_{r-2} at a fixed angle α. It is also the envelope of spheres whose centres lie on a hyperboloid H_{r-2}, and which intersect an $n-r$-dimensional sphere K_{n-r} at a constant angle α'.*

It remains to discuss the nature of the loci of double points on the cyclide. For real cyclides the equation of the principal sphere

$$\sum_1^{n-2}X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{\left[\frac{1}{1-a^2}-\frac{a^2}{k^2(1-a^2)^2}\right]\left[\frac{a^2}{1-a^2}-\frac{a^2}{k^2(1-a^2)}\right]}{\frac{a^2}{k^2(1-a^2)}}$$

is also real. Let the centre on the X_{n-1}-axis $-\frac{a}{k(1-a^2)}$ be inside the ellipsoid and between the two sheets of the hyperboloid. Σ_{r-3} is then real, the focal locus K_{n-r} is then imaginary and K_{r-2} is real, as is also the principal sphere. The sphere Σ_{n-r} is imaginary and the cyclide has one real locus of double points only.

Let the centre C be between the foci of the hyperboloid and the ellipsoid. The principal sphere is imaginary, since the two factors in the numerator of R^2 have opposite signs. The focal spheres K_{r-2} and K_{n-r} are real since $k^2(1-a^2)-a^2$ is positive. The cyclide has no real double locus.

Finally, when the centre C is outside of the ellipsoid, the principal sphere is again real while Σ_{r-3} is imaginary. The sphere Σ_{n-r-1} is real and the cyclide has one real locus of double points.

§ 8. *Transformations.*

If we transform the cyclide by an inversion whose pole is on one of the focal loci, say K_{r-2}, it is transformed into a torus. In fact, the focal sphere K_{r-2} is transformed into an $r-2$-flat, and the spheres S_h which, as may easily be proved, intersects K_{r-2} orthogonally are transformed into spheres whose centres lie on this flat.* The transform of the cyclide is therefore the envelope of spheres whose centres lie on the $r-2$-flat and touch a given sphere, namely a transform of one of the spheres S_e. By revolving this latter sphere about the flat we get a torus, the transform of the cyclide. If the pole lies on the sphere K_{n-r} this sphere is transformed into an $n-r$-flat and the spheres S, which are orthogonal to K_{n-r}, are transformed into spheres whose centres lie on the $n-r$-flat. These spheres touch any one of the transforms of S_h and the new surface is therefore obtained by revolving the sphere S_h about the $n-r$-flat as an axis. The surface is therefore a torus. *There are* ∞^{n-2} *inversions which transform a cyclide into a torus.* If, in particular, the pole lies on the sphere Σ_{n-r-1} (or on Σ_{r-3}) we obtain a cone of revolution. There are therefore ∞^{n-r-1} inversions which will transform a cyclide of type $r \leq \frac{n+2}{2}$ into a cone of revolution.

Since the cyclide has only two focal sheets, the determinant in § 3, note, has only two roots, σ_1 and σ_2, of multiplicity $r-2$ and $n-r$, respectively. There exist, therefore, corresponding to $R_1 \infty^{r-3}$ principal directions on the surface which lie in a space S_{n-r}; and corresponding to R_2 there are ∞^{n-r-1} principal directions lying in a space S_{r-2}.† Thus we found that from any point A on the locus of double points Σ_{r-3} pass ∞^{n-r-1} isotropic lines forming an isotropic cone of revolution whose elements are lines of curvature. Through any point P not on Σ_{r-3} or Σ_{n-r-1} pass two pencils of ∞^{n-r-1} and ∞^{r-3} circles, respectively. The circles through P generate two spherical spaces S_{n-r} and

* As a consequence we have a new mode of generation: *a cyclide of the fourth order is generated in two ways by spheres whose centres lie on a hyperboloid of revolution H_{r-2} (or an ellipsoid of revolution E_{n-r}) and which intersect a sphere K_{r-2} (or K_{n-r}) at right angles.*

† See Bianchi, *Lezioni*, Vol. I, p. 369. Seconda edizione.

S_{r-2} which intersect at right angles, as may easily be proved by transforming the cyclide into a cone of revolution.

We consider two special cases. If $k=0$, Σ_{r-3} is the imaginary sphere $\sum_{n-r+1}^{n-2} X_i^2=-1$, $X_1=\ldots=X_{n-r}=X_{n-1}=0$ and Σ_{n-r-1} becomes the unit sphere $\sum_1^{n-r} X_i^2=1$, $X_{n-r+1}=X_{n-r+2}=\ldots=X_{n-1}=0$. The cyclide is symmetrical with respect to all the coordinate planes.

If Σ_{r-3} is the null-sphere whose centre is on the vertex of the focal hyperboloid we have $k=\dfrac{1}{\sqrt{1-a^2}}$. The focal sphere K_{n-r} becomes the point-sphere and Σ_{n-r-1} an imaginary sphere with radius $\mp\dfrac{\sqrt{a^2-1}}{a}$ which is immersed in the directrix-space of the focal ellipsoid.

An inversion with pole on Σ_{n-r-1} or Σ_{r-2} will transform the cyclide into a cylinder of revolution. The cyclide will have a single real double locus in finite space as in the general case.

§ 9. *Cyclides of the Third Order.*

If $a=1$ we have a type of cyclides of the third order, the parabolic type, obtained in § 6 (49′). The locus of point-spheres on the surface are gotten by equating to zero the radii of the two sets of spheres (50) which we shall denote by S_{ρ_1} and S_{ρ_2}. We have then

$$1+2\sum_1^{n-r}\rho_i^2=k,\quad 1+2\sum_{n-r+1}^{n-2}\rho_i^2=-k. \tag{66}$$

Substituting in equations (50) we have the two point-spheres

$$\left.\begin{aligned}&\sum_1^{n-r-1}(X_i+2\rho_i)^2+\left(X_{n-r}\pm2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2}\right)^2+\sum_{n-r+1}^{n-2}X_i^2+\left(X_{n-1}+\frac{2-k}{2}\right)^2=0,\\&\sum_1^{n-r}X_i^2+\sum_{n-r+1}^{n-3}(X_i-2\rho_i)^2+\left(X_{n-2}\mp2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}^{n-2}\rho_i^2}\right)^2\\&\qquad\qquad+\left(X_{n-1}-\frac{2+k}{2}\right)^2=0.\end{aligned}\right\} \tag{67}$$

The loci of point-spheres are therefore:

$$\left.\begin{aligned}&X_1=-2\rho_1,\ \ldots,\ X_{n-r-1}=-2\rho_{n-r-1},\quad X_{n-r}=\mp2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2},\\&\qquad X_{n-r+1}=\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{k}{2}-1,\\&X_1=\ldots=X_{n-r}=0,\quad X_{n-r+1}=2\rho_{n-r+1}=\ldots=X_{n-3}=2\rho_{n-3},\\&\qquad X_{n-2}=\pm2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}\rho_i^2},\quad X_{n-1}=1+\frac{k}{2}.\end{aligned}\right\} \tag{68}$$

Eliminating the ρ's we have the spheres

$$\left.\begin{aligned}&\sum_{1}^{n-r} X_i^2= \ \ 2(k-1),\ \ X_{n-r+1}=\ldots\ldots=X_{n-2}=0,\ \ X_{n-1}=\frac{k}{2}-1,\\ &\sum_{n-r+1}^{n-2} X_i^2=-2(k+1),\ \ X_1=\ldots\ldots=X_{n-r}=0,\ \ X_{n-1}=\frac{k}{2}+1,\end{aligned}\right\}\quad(69)$$

which are loci of double points on the surface. If $k>1$ or <-1, one sphere is imaginary while the other is real, and if $k<1$ or >-1, the spheres are both imaginary. We shall denote these spheres by Σ_{p_1} and Σ_{p_2}. All the spheres S_{p_1} pass through Σ_{p_2}, and all spheres S_{p_2} pass through Σ_{p_1}. Hence the cyclides of the third order in S_{n-1} is the envelope of spheres $\begin{Bmatrix}S_{p_1}\\S_{p_2}\end{Bmatrix}$ having their centres on a paraboloid of revolution $\begin{Bmatrix}P_1\\P_2\end{Bmatrix}$ and passing through a spherical spread $\begin{Bmatrix}\Sigma_{p_2}\\\Sigma_{p_1}\end{Bmatrix}$ on the paraboloid $\begin{Bmatrix}P_2\\P_1\end{Bmatrix}$. The locus D_{n-4} generated by the isotropic line which joins a point A of Σ_{p_1} to a point B on S_{p_2} belongs to an isotropic developable Δ_{n-3}, and the cyclide is inscribed in this developable along the generators of D_{n-4}, which are lines of curvature of the surface. The focal spheres K_{p_1} and K_{p_2} are

$$\left.\begin{aligned}&\sum_{n-r+1}^{n-2} X_i^2+[X_{n-1}+\tfrac{1}{2}(2-k)]^2=2(1-k),\ \ X_1=X_2=\ldots\ldots=X_{n-r}=0,\\ &\Sigma\ X_i^2+[X_{n-1}-\tfrac{1}{2}(2+k)]\ \ =2(1+k),\ \ X_{n-r+1}\ =\ldots\ldots=X_{n-2}=0.\end{aligned}\right\}\quad(70)$$

These spheres are tangents to the focal paraboloids P_1 and P_2 along the two loci of double points, respectively. Analogous to the theorem for cyclides of the fourth order we may prove that

A cyclide of the third order in S_{n-1} is in two ways the envelope of spheres $\begin{Bmatrix}S_{p_1}\\S_{p_2}\end{Bmatrix}$ whose centres lie on a paraboloid of revolution $\begin{Bmatrix}P_1\\P_2\end{Bmatrix}$ and which intersect a fixed sphere $\begin{Bmatrix}K_2\\K_1\end{Bmatrix}$ at a constant angle.

Since the spheres $\begin{Bmatrix}S_1\\S_2\end{Bmatrix}$ intersect the focal sphere $\begin{Bmatrix}K_1\\K_2\end{Bmatrix}$ at right angles we may say:

A cyclide of the third order is in two ways the envelope of spheres $\begin{Bmatrix}S_{p_1}\\S_{p_2}\end{Bmatrix}$ whose centres lie on a paraboloid of revolution $\begin{Bmatrix}P_1\\P_2\end{Bmatrix}$ and intersect a fixed sphere $\begin{Bmatrix}K_1\\K_2\end{Bmatrix}$ at right angles.

§ 10. *Surfaces in S_{n-1} Analogous to Moulding Surfaces with Cylindrical Directrix in 3-Space.*

If we put $a=0$ in (29) the spherical representation of the lines of curvature will consist of a system of parallels and meridians on the Gaussian sphere; the corresponding surfaces are then analogous to a certain kind of moulding surfaces in 3-space, viz.: those having a cylindrical directrix. The surface is the envelope of the $n-2$-flat

$$\sum_1^{n-1} x_i X_i = \Sigma \Phi_i \sqrt{E_i}, \tag{71}$$

the x's and the E's having the following values:

$$\left.\begin{array}{ll} x_1=\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_1, & x_2=\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_2\cos\theta_1, \\ \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, \\ x_{n-r}=\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}, & x_{n-r+1}=\cos\theta_{n-r+1}\ldots\cos\theta_{n-2}, \\ \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, \\ x_{n-2}=\cos\theta_{n-2}\sin\theta_{n-r+1}, & x_{n-1}=\cos\theta_{n-r}\sin\theta_{n-2}, \end{array}\right\} \tag{72}$$

$$\left.\begin{array}{l} E_1=\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_2, \quad E_2=\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_3, \\ \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \\ E_{n-r}=\sin^2\theta_{n-2}, \quad E_{n-r+1}=\cos^2\theta_{n-2}, \quad E_{n-r+2}=\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}, \ldots, \\ E_{n-3}=\cos^2\theta_{n-2}\cos^2\theta_{n-4}\ldots\cos^2\theta_{n-r+1}, \quad E_{n-2}=1. \end{array}\right\} \tag{73}$$

Differentiating (73) partially with respect to the θ's, and solving for the X's, we have the equations of the surface in the form:

$$\left.\begin{array}{l} X_1=\sin\theta_1\ldots\sin\theta_{n-r}\phi_{n-2} \\ \qquad +\sin\theta_1\ldots\sin\theta_{n-r-1}\phi_{n-r}+\ldots+\sin\theta_1\phi_2+\phi_1, \\ X_2=\cos\theta_1\sin\theta_2\ldots\sin\theta_{n-r}\phi_{n-2} \\ \qquad +\cos\theta_1\sin\theta_2\ldots\sin\theta_{n-r-1}\phi_{n-r}+\ldots+\cos\theta_1\phi_2+\psi_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-r}=\cos\theta_{n-r-1}\sin\theta_{n-r}\phi_{n-2}+\cos\theta_{n-r-4}\phi_{n-r}+\psi_{n-r-1}, \\ X_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\psi_{n-2} \\ \qquad +\cos\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3}, \\ X_{n-r+2}=\sin\theta_{n-3}\cos\theta_{n-4}\ldots\cos\theta_{n-r+1}\psi_{n-2} \\ \qquad +\sin\theta_{n-3}\cos\theta_{n-4}\ldots\cos\theta_{n-r-2}\psi_{n-r+2}+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-2}=\sin\theta_{n-r+1}\psi_{n-2}+\phi_{n-r+1}, \quad X_{n-1}=\cos\theta_{n-r}\phi_{n-2}+\psi_{n-r}, \end{array}\right\} \tag{74}$$

where

$$\phi_i=\Phi_i\sin\theta_i+\Phi_i'\cos\theta, \quad \psi_i=\Phi_i\cos\theta_i-\Phi_i'\sin\theta_i.$$

The linear element of the surface is

$$ds^2 = \Sigma b_i d\theta_i,$$

where the b's have the values:

$$\left.\begin{aligned}
b_1 &= [\sin\theta_2 \ldots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_2 \ldots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \ldots + \phi_2 + \Phi_1 + \Phi_1'']^2, \\
b_2 &= [\sin\theta_3 \ldots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_3 \ldots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \ldots + \phi_3 + \Phi_2 + \Phi_2'']^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
b_{n-r} &= [\phi_{n-2} + \Phi_{n-r} + \Phi''_{n-r}]^2, \quad b_{n-r+1} = [\psi_{n-2} + \Phi_{n-r+1} + \Phi''_{n-r+1}]^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
b_{n-3} &= [\cos\theta_{n-4} \ldots \cos\theta_{n-r+1}\psi_{n-2} + \cos\theta_{n-4} \ldots \cos\theta_{n-r+2}\psi_{n-r+1} \\
&\qquad + \ldots + \psi_{n-4} + \Phi_{n-3} + \Phi''_{n-3}]^2, \\
b_{n-2} &= [\Phi_{n-2} + \Phi''_{n-2}]^2,
\end{aligned}\right\} \quad (75)$$

from which it is seen that the lines $\theta_1 = c_1$, $\theta_2 = c_2$, $\ldots$, $\theta_{n-3} = c_{n-3}$ are geodetic.

The moulding surfaces (74) have certain geometric properties which we shall now discuss.* There are $n-3$ types of these surfaces obtained by giving to r the successive values $3, 4, \ldots, n-1$; however, these types are not distinct, in fact, the types for which $r=k$ and $r=n-k+2$ will not yield essentially different surfaces as is seen from equations (72) and (74). If n is odd there are $\frac{n-3}{2}$ distinct types, and $\frac{n-2}{2}$ if n is even.

Consider the surface V_{n-r},

$$\left.\begin{aligned}
\xi_1 &= \sin\theta_1 \ldots \sin\theta_{n-r-1}\phi_{n-r} + \ldots + \sin\theta_1\phi_2 + \phi_1, \\
\xi_2 &= \cos\theta_1 \sin\theta_2 \ldots \sin\theta_{n-r-1}\phi_{n-r} + \ldots + \cos\theta_1\phi_2 + \psi_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\xi_{n-r} &= \cos\theta_{n-r-1}\phi_{n-r} + \psi_{n-r-1}, \quad \xi_{n-1} = \psi_{n-r}, \quad \xi_{n-r+1} = \xi_{n-r+2} = \ldots = \xi_{n-2} = 0,
\end{aligned}\right\} \quad (76)$$

immersed in the space $\xi_{n-r+1} = \ldots = \xi_{n-2} = 0$; it is a moulding surface of type $r'=3$ in a space S_{n-r+1}. Consider also the cylindrical surface C_{n-r+1} obtained by constructing the normals to the space S_{n-r+1} at every point of V_{n-r}; the direction cosines of these normals to V_{n-r} lying in S_{n-r+1} are:

$$\sin\theta_1 \ldots \sin\theta_{n-r}, \ldots, \cos\theta_{n-r-1}\sin\theta_{n-r}, 0, 0, \ldots, 0, \cos\theta_{n-r},$$

and these are also the direction cosines of a normal to C_{n-r+1} at any point of

*A special case of the surfaces (74) has been obtained by Umberto Sbrana in an article entitled "I sistemi ciclici nello spazio euclidio ad n dimensione," *Rendiconti del Circolo Matematico di Palermo*, Tomo, XIX, pp. 258-290. The surface is one of type $r=3$, the only distinct type that can exist in 4-space ($n=5$) since $\frac{n-3}{2}=1$ for $n=5$.

the generator passing through the point $\theta_1 \ldots \theta_{n-r}$. We now construct a surface V_{r-2}:

$$\left.\begin{array}{l} \zeta_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\psi_{n-2}+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3}, \\ \zeta_{n-r+2}=\sin\theta_{n-3}\ldots\cos\theta_{n-r+1}\psi_{n-2}+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \zeta_{n-2}\;=\sin\theta_{n-r+1}\psi_{n-2}+\phi_{n-r+1},\;\; \zeta_{n-1}=\phi_{n-2},\;\; \zeta_{n-r}=\ldots=\zeta_1=0, \end{array}\right\} \quad (77)$$

which is a moulding surface in space S_{r-1} of type $r'=3$. This space has the maximum of perpendicularity with the space S_{n-r+1}.

A moulding surface V_{n-2} in S_{n-1} may now be constructed geometrically as follows: In the space S_{n-r+1} of the ambient space S_{n-1} we construct a moulding surface V_{n-r} of type $r'=3$, and a cylindrical surface C_{n-r+1} which is the locus of normals to S_{n-r+1} erected at all points of V_{n-r}. We then construct a moulding surface V_{r-2} in space S_{r-1} which has the maximum of perpendicularity with S_{n-r+1} with a convenient orientation of axes $\zeta_{n-r+1}, \ldots, \zeta_{n-1}$. Let this space take all the ∞^{n-r} positions that are obtained by making a certain one of the given axes coincide with a generator of the cylinder C_{n-r+1}, and the remaining $r-2$-axes with the $r-2$ oriented directions normal to C_{n-r+1} in the point in which the generator meets V_{n-r}. The surface generated will be a moulding surface V_{n-2}. This generation may be described as a "rolling" of S_{r-1} on the cylinder C_{n-r+1}, the points of V_{r-2} in S_{r-1} generating the surface V_{n-2}. A moulding surface in S_{n-1} of type $r=3$ is thus generated by a plane S_2 rolling on a cylinder C_{n-2}, the points of the curve V_1 in the plane generating the surface. Hence it follows that *starting with a moulding surface in 3-space* (with cylindrical directrix) *the moulding surfaces of type $r'=3$ may be generated in the next higher spaces up to, say, S_{n-r+1} and S_{r-1}, and finally V_{n-2} by "rolling" the space S_{r-1} on the cylinder constructed with V_{n-r} as a base. S_{r-1} must have maximum perpendicularity with S_{n-r+1}. The surface generated by the points of V_{r-2} is of type r.*

If in (74) we put $\theta_{n-2}=c_{n-2}$ we get two moulding surfaces of type $r'=3$, one immersed in a space $S(X_1, X_2, \ldots, X_{n-r}, X_{n-1})$, and the other in a space $S(X_{n-r+1}, \ldots, X_{n-2})$, both being curved varieties on V_{n-2}. Starting with a moulding surface in 3-space we may construct each one of these by rolling a plane in which is drawn a profile-curve $\xi_i=\phi_i$, $\eta_i=\Psi_i$, over a cylindrical directrix C_i whose base is a moulding surface V_{i-1} of type $r'=3$ in the next lower space; since the profile-curve is plane it will generate a set of plane lines of curvature on V_i. The lines of curvature on V_i remain lines of curvature on V_{i+1} in the next higher space. But the lines of curvature θ_{n-2} are also plane on V_{n-2}, hence they are all plane.

The $n-2$ focal sheets are all distinct; in fact, if we calculate the values of the principal radii we find:

$$R_1=\frac{-\sqrt{b_1}}{\sin\theta_2\ldots\sin\theta_{n-r}\sin\theta_{n-2}},\ \ldots,\quad R_{n-r}=\frac{-\sqrt{b_{n-r}}}{\sin\theta_{n-2}},$$

$$R_{n-r+1}=\frac{-\sqrt{b_{n-r+1}}}{\cos\theta_{n-2}},\ \ldots,\quad R_{n-3}=\frac{-\sqrt{b_{n-3}}}{\cos\theta_{n-r+1}\ldots\cos\theta_{n-4}\cos\theta_{n-2}},$$

$$R_{n-2}=-\sqrt{b_{n-2}}.$$

§ 11. *Surfaces of Revolution.*

If $\Phi_1=\Phi_2=\ldots=\Phi_{n-r}=0$, we get a species of surfaces of revolution in S_{n-1}. If we introduce the arc of the curve $\zeta_{n-2}=\psi_{n-2}$, $\zeta_{n-1}=\phi_{n-2}$ as a new parameter u instead of θ_{n-2}, the equations of the surface may be written

$$X_1=\sin\theta_1\ldots\sin\theta_{n-r}U,\ \ldots,\quad X_{n-r}=\cos\theta_{n-r+1}\sin\theta_{n-r}U,$$

$$X_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\cos\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3},$$

$$X_{n-r+2}=\sin\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\sin\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$X_{n-2}\doteq\sin\theta_{n-r+1}\int_{u_0}^{u_1}\sqrt{1-U'^2}+\phi_{n-r+1},\quad X_{n-1}=\cos\theta_{n-r}U,$$

generated by the moulding surface V (equations 76), turning about the coordinate flat $X_1=X_2=\ldots=X_{n-r}=X_{n-1}=0$ as an axis. Consider the variety $\theta_{n-r+1}=c_{n-r+1}, \ldots, \theta_{n-2}=c_{n-3}$, $u=c$ on the surface; it is a sphere of constant radius $U(c)$ and its centre lies on the axial space $X_1=\ldots=X_{n-r}=X_{n-1}=0$. Let c be fixed while $\theta_{n-r+1}, \ldots, \theta_{n-3}$ vary; the locus of centres is a moulding surface in $S(X_{n-r+1}, \ldots, X_{n-2})$ of type $r'=3$. Any line traced on the ∞^{r-2} spheres $u=c$, where c has any value, is a line of curvature on the surface. Since $R_1=R_1=\ldots=R_{n-r}$ the surface has $r-1$ distinct focal sheets. A surface of revolution of type $r=3$ has $r-1=2$ distinct focal sheets.*

§ 12. *Transformations of the Gaussian Sphere. Orthogonal Systems.*

From equations (37) it follows that given the spherical representation (29) the function $W=\Sigma\Phi_i\sqrt{E_i}$ defines all surfaces whose lines of curvature are plane in all systems. The function W satisfies the following system of differential equations:

$$\frac{\partial^2 W}{\partial\theta_i\partial\theta_k}=\frac{\partial\log\sqrt{E_k}}{\partial\theta_i}\frac{\partial W}{\partial\theta_k}+\frac{\partial\log\sqrt{E_i}}{\partial\theta_k}\frac{\partial W}{\partial\theta_i};\qquad i,k=1,2,\ldots,n-2,\quad i\neq k.\tag{78}$$

*A surface of this kind ($r=3$) in 4-space has been constructed by Sbrana, *loc. cit.*, p. 287.

If now we put

a) $$\lambda=\Delta_1'W+(\theta_{n-1}+W)^2=\sum\frac{1}{E_i}\left(\frac{\partial W}{\partial\theta_i}\right)^2+(\theta_{n-1}+W)^2,$$

where θ_{n-1} denotes an arbitrary constant, we have the following application of a general theorem proved by Sbrana: *

There exists a transformation of the Gaussian sphere into itself by means of which the differential form (30) *is transformed into one of the form*

$$d\sigma^2=\lambda^2\sum_1^{n-2}\left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial\theta_i}}{\dfrac{\partial W}{\partial\theta_i}}\right]^2 E_i d\theta_i^2, \quad (\theta_{n-1}=\text{arbitrary constant.})$$

To a point x_i corresponds a point x_i' by means of the equations

b) $$x_i'=x_i-\rho\frac{\theta_{n-1}+W}{\Psi}\left[(\theta_{n-1}+W)x_i+\rho\nabla'(x_i,W)\right],$$

where

c) $$\Psi=\int\sum R_i\frac{\partial W}{\partial\theta_i}d\theta_i \text{ and } -\frac{2\Psi}{\rho}+\Delta'W+(\theta_{n-1}+W)^2=0.$$

The corresponding $n-1$-tuple orthogonal system has for linear element

$$ds^2=\Sigma d\xi_i^2=\rho^2 d\theta_{n-1}^2+\lambda^2\sum_1^{n-2}\left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial\theta_i}}{\dfrac{\partial W}{\partial\theta_i}}\right]^2 E_i d\theta_i^2,$$

the coordinates ξ_i being given by the equations:

$$\xi_i=X_i+\rho(\theta_{n-1}+W)x+\rho\nabla'(x_i,W),$$

the symbol ∇' in (b) *being equivalent to the summation*

$$\sum\frac{1}{E_i}\frac{\partial x_i'}{\partial\theta_i}\frac{\partial W}{\partial\theta_i}.$$

All the cyclic systems normal to the surfaces whose tangential equation is given by (37) may be obtained by quadrature. In particular, the cyclic systems normal to the surfaces (74) are gotten by integrating the equation

$$d\Psi=\sum_1^{n-2}R_i\frac{\partial W}{\partial\theta_i}d\theta_i,$$

* Sbrana, *loc. cit.*, p. 227. An application to a very special case is given on p. 282.

where the R's have the values given at the end of §10 and $W=\Sigma\Phi_i\sqrt{E_i}$, (eq. 71). For the general surface (37) the calculation of the R's involves considerable work.

II.

§1. *Transformations. Asymptotic Lines.*

Consider the $n-2$-spread (6),

$$\left.\begin{aligned} X_1 &= \frac{(\alpha_1+\beta_1)}{2}X_{n-1}-\frac{1}{2}(F_{\alpha_1}+F_{\beta_1}),\\ X_2 &= i\frac{(\alpha_1-\beta_1)}{2}X_{n-1}+\frac{i}{2}(F_{\alpha_1}-F_{\beta_1}),\\ &\dots\dots\dots\dots\dots\dots\dots\dots,\\ &\dots\dots\dots\dots\dots\dots\dots\dots,\\ X_{n-1} &= \frac{\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{aligned}\right\} \qquad (79)$$

If, as was done in our former paper (A, p. 203) we consider F, α_i, β_i, F'_{α_i}, F'_{β_i} as the surface-elements of a space S_{n-1}, the above system may be looked upon as a contact-transformation which carries the surface-elements of S_{n-1} into those of $\bar{S}_{n-1}$. To the lines of curvature on a surface in S_{n-1} correspond a set of conjugate lines on the transform to which we have given the name Euler's lines or E-lines. These lines have the property of being transformed into asymptotic lines on a surface in a space $\bar{\bar{S}}_{n-1}$ by means of Euler's transformation

$$x_i=F'_{\alpha_i},\quad y_i=\beta_i,\quad p_i=-\alpha_i,\quad q_i=F'_{\beta_i},\quad z=F-\Sigma\alpha_iF'_{\alpha_i}. \qquad (80)$$

These E-lines, while they have been implicitly used by several authors,* have never been considered from the view-point of the theory of contact-transformations before the author's treatment of them in an article in *American Transactions*, Vol. VI, pp. 450–471. Since Euler's transformation applies to odd space the E-lines do not exist on a surface in even-dimensional space, and, as a consequence, the asymptotic lines exist in odd space only.

The tangential equation of a surface in an even or odd space S_{n-1} may be written as before:

$$2y_1X_1+2y_2X_2+\dots\dots+(1-\Sigma y_i^2)X_{n-1}+F=0, \qquad \text{(I)}$$

and, if S_{n-1} is an odd space,

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\dots\dots+(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0, \qquad \text{(II)}$$

* Darboux, *Leçons*, Vol. I, pp. 200–202, deuxième édition, Vol. IV, p. 171.

which latter form is reducible to the first by means of the transformation

$$\alpha_1+\beta_1=2y_1,\quad i(\alpha_1-\beta_1)=2y_2,\ \ldots,\quad (1-\Sigma\alpha_i\beta_i)=1-\Sigma y_i^2.$$

The lines of curvature of the surface M_{n-2} defined as the envelope of the tangent planes (II) are given by the system of differential equations (A, p. 211),

$$\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha k}}{dF_{\beta_{\frac{n-2}{2}}}},\quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta i}}{dF_{\beta_{\frac{n-2}{2}}}},\qquad \begin{array}{l} i=1,2,\ldots,\frac{n-4}{2},\\ k=1,2,\ldots,\frac{n-2}{2},\end{array}$$

to which in the case of (I) corresponds the system

$$\frac{dy_i}{dy_{n-2}}=\frac{dF'_{y_i}}{dF_{n-2}},\qquad i=1,2,\ldots,n-3, \tag{81}$$

and we have seen (§ 2) that if the lines of curvature on a surface are *coordinate lines*, the *y's*, considered as functions of the ρ's, must form a completely orthogonal system in an $n-2$-space, and that F is a particular or general solution of the system of differential equations (12). The contact-transformation corresponding to (6) now takes the simple form:

$$X_i=\frac{y_i(\Sigma y_iF'_{y_i}-F)}{1+\Sigma y_i^2}-\tfrac{1}{2}F'_{y_i},\quad X_{n-1}=\frac{\Sigma y_iF'_{y_i}-F}{1+\Sigma y_i^2}, \tag{82}$$

which transforms spheres into paraboloids of the form:

$$F=a\Sigma y_i^2+\Sigma b_iy_i+c.$$

The Theorem II, (§ 2), is therefore true in any space, and the surfaces obtained by the application of the theory exist in any space, even or odd.

If we consider an odd space $\overline{S}_{n-1}$ and a spread $F=F(\alpha_i,\beta_i)$, Theorem II may be stated thus:

If, on a surface in $\overline{S}_{n-1}$, a system of curves are coordinate E-lines, the coordinates F, α_i, β_i satisfy the system of differential equations

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}-\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}=0, \tag{12a}$$

and α_i, β_i satisfy the $\dfrac{n-2\cdot n-3}{2}$ relations

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)=0,\qquad k,k'=1,2,\ldots,n-2,$$

and the surface F is found by taking any particular or general solution of (12a).

If we transform to $\overline{\overline{S}}_{n-1}$, using Euler's transformation, we have the following

THEOREM III. *If on a surface in $\bar{\bar{S}}_{n-1}$ a system of curves are coordinate asymptotic lines, the coordinates y_i and p_i of the surface-elements on the surface must satisfy the differential equations* (12a) *and also the* $\frac{n-2 \cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial y_i}{\partial \rho_{k'}}\frac{\partial p_i}{\partial \rho_k}+\frac{\partial y_i}{\partial \rho_k}\frac{\partial p_i}{\partial \rho_{k'}}\right)=0.$$

This theorem solves the problem of *Lelieuvre* for general odd space. A system of solutions y_i, p_i having been found, q_i and x_i may be found by quadratures and hence also z. It should be observed that $z-\Sigma x_i p_i$ is a particular solution of (12a).

In order to find surfaces with coordinate asymptotic lines it will be most convenient to find in S_{n-1} corresponding surfaces with coordinate lines of curvature. The chief difficulty which we encounter is the fact, pointed out before, that to real elements in S_{n-1} correspond in general imaginary elements in $\bar{\bar{S}}_{n-1}$ with the obvious result that the surface, if real in S_{n-1}, has often an imaginary transform in $\bar{\bar{S}}_{n-1}$ and vice versa. If, however, we have a real surface in $\bar{S}_{n-1}$, *i. e.*, if F considered as a function of α_i, β_i, is real, the corresponding surface in $\bar{\bar{S}}_{n-1}$ is real.

§ 2. Consider the surfaces for which

$$F=2\phi_1(\rho_1)+2\phi_2(\rho_2)+\ldots.+2\phi_{n-2}(\rho_{n-2})+2C$$

obtained on p. 7. If we transform by Euler's transformation we get the surface

$$\left.\begin{aligned} x_i&=\phi'_{2i-1}+i\phi'_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i},\\ z&=2\Sigma\phi_i(\rho_i)-\Sigma(\rho_{2i-1}-i\rho_{2i})(\phi'_{2i-1}+i\phi'_{2i})+2C, \end{aligned}\right\} \quad (83)$$

which is real if ϕ_{2i} is an even function of ρ_{2i}.

As a particular case let us take a cyclide of the third order and type $r=\frac{n-2}{2}$ obtained by giving to the ϕ's the following values:*

$$\phi_{2i-1}=\frac{k+2}{2}\rho^2_{2i-1}, \quad \phi_{2i}=\frac{k-2}{2}\rho^2_{2i}, \quad 2C=\frac{k}{2}.$$

We get the real surface

$$x_i=\frac{k+2}{2}\rho_{2i-1}+i\frac{k-2}{2}\rho_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i}, \quad z=2i\Sigma\rho_{2i-1}\rho_{2i}+\frac{k}{2}, \quad (84)$$

that is, the paraboloid

$$z+\frac{1}{2}\sum_1^{\frac{n-2}{2}}\left[x_i^2+\frac{k^2-4}{4}y_i^2-kx_iy_i\right]=\frac{k}{2}. \quad (84')$$

* This surface differs from (49') only in orientation of axes.

This surface is generated in two different ways by self-dual flats corresponding to the two sets of curvature-spheres which generate its transform in S_{n-1}. These flats are evidently:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2} y_i + 2\rho_{2i-1}, & x_i &= \frac{k+2}{2} y_i - 2i\rho_{2i}, \\ z &= 2\Sigma[\rho_{2i-1} y_i - \rho_{2i-1}^2] + \frac{k}{2}, & z &= 2i\Sigma[\rho_{2i} y_i - i\rho_{2i}^2] + \frac{k}{2}. \end{aligned}\right\} \quad (85)$$

If we introduce the parameters α_i, β_i instead of the ρ's in these equations we see that they are real flats. The asymptotic lines are to a certain extent indeterminate. In fact, the flats I and II lie entirely in the surface (84′), and through any point on the surface pass two flats so that any curve immersed in either one of them is an asymptotic curve. Through any point will pass $\infty^{\frac{n-4}{2}}$ asymptotic directions.

To the flat

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d, \quad (86)$$

corresponds a sphere (A, p. 217)

$$\left(X_1 + \frac{b_1 + c_1}{2}\right)^2 + \left(X_2 - i\frac{b_1 - c_1}{2}\right)^2 + \ldots . + \left(X_{n-1} - \frac{a-d}{2}\right)^2 = \left(\frac{a+d}{2}\right)^2,$$

but to a sphere corresponds two flats since $R = \pm\frac{a+d}{2}$. We shall call the flat for which $R = +\frac{a+d}{2}$ the *positive correspondent* and the one for which $R = -\frac{a+d}{2}$ the *negative correspondent*. To point-spheres correspond flats for which $a = -d$, that is, the two correspondents coincide. Any pair of $+$ and $-$ correspondents in the same set are said to be *conjugate* with respect to the flat-complex $a+d=0$ (A, p. 217). Comparing now equations (51), p. 20, with (85) and (86), it appears that the flats of the first set are negative correspondents, and those of the second set are positive correspondents. To the nodal loci on the cyclide correspond two sets of flats on the paraboloid (84) belonging to the flat-complexes $a+d=0=\frac{k-2}{2} - 2\Sigma\rho_{2i-1}^2 + \frac{k}{2}$, and $a+d=0 = \frac{k+2}{2} + 2\Sigma\rho_{2i}^2 + \frac{k}{2}$:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2} y_i + 2\rho_{2i-1}, & x &= \frac{k+2}{2} y_i - 2i\rho_{2i}, \\ z &= 2\Sigma\rho_{2i-1} y_i - \frac{k}{2} + 1, & z &= 2\Sigma\rho_{2i} y_i - \frac{k}{2} - 1, \\ 2\Sigma\rho_{2i-1}^2 &= (k-1), & 2\Sigma\rho_{2i}^2 &= -(1+k). \end{aligned}\right\} \quad (87)$$

They are the common generators of the quadric and the two cylinders,

$$\Sigma\left(x_i-\frac{k-2}{2}\,y_i\right)^2=2(k-1),\quad \Sigma\left(x_i-\frac{k+2}{2}\,y_i\right)^2=-2(1+k). \tag{88}$$

If the cyclide is of type $r\neq\frac{n-2}{2}$, we give to the ϕ's the following values:

$$\phi_{2i-1}=\frac{k+2}{4}\rho^2_{2i-1},\quad \phi_{2i}=\frac{k-2}{4}\rho^2_{2i},\quad \phi_{2r+t}=\frac{k-2}{4}\rho^2_{2r+t},\qquad \begin{array}{l} i=1,2,\ldots,r,\\ t=1,2,\ldots,r.\end{array}$$

The transform of the cyclide is:

$$\left.\begin{array}{lll} x_j=\frac{k+2}{2}\rho_{2j-1}+i\frac{k-2}{2}\rho_{2j}, & y_i=\rho_{2i-1}+i\rho_{2i}, & \begin{array}{l} j=1,2,\ldots,r,\\ i=1,2,\ldots,\frac{n-2}{2},\end{array}\\ x_{r+s}=\frac{k-2}{2}[\rho_{2(r+s)-1}+i\rho_{2(r+s)}]y_{r+s}, & z=i\Sigma\rho_{2j}\rho_{2j-1}+\frac{k}{2}, & \\ & & s=1,2,\ldots,\frac{n-2-2r}{2}. \end{array}\right\} \tag{89}$$

Eliminating the ρ's we get the parabolic surface of $\frac{n-2}{2}+r$ dimensions

$$z=-\frac{1}{2}\sum_1^r\left[x_j^2+\frac{k^2-4}{4}y_j^2-kx_jy_j\right]+\frac{k}{2};\quad x_{r+s}=\frac{k-2}{2}y_{r+s},$$
$$s=1,2,\ldots,\frac{n-2}{2}-r. \tag{90}$$

As in the preceding special case the two sets of spheres generating the cyclide are transformed into two sets of flats:

$$\text{I}\begin{cases} x_j=\frac{k-2}{2}y_i+2\rho_{2j-1},\\ x_{r+s}=\frac{k-2}{2}y_{r+s},\\ z=2\sum_1^r\rho_{2j-1}y_j-2\sum_1^r\rho^2_{2j-1}+\frac{k}{2}. \end{cases}$$

$$\text{II}\begin{cases} x_j=\frac{k+2}{2}y_j-2i\rho_{2j},\\ x_{r+s}=\frac{k+2}{2}y_{r+s}-2(\rho_{2(r+s)-1}+i\rho_{2(r+s)}),\\ z=2i\Sigma\rho_{2j}y_j-2\Sigma(\rho_{2(r+s)-1}-i\rho_{2(r+s)})y_{r+s}+2\sum_1^{\frac{n-2}{2}}\rho^2_{2j}+2\sum_{r+1}^{\frac{n-2}{2}}\rho^2_{2j-1}+\frac{k}{2},\\ \qquad\left(j=1,2,\ldots,r,\quad s=r+1,\ldots,\frac{n-2}{2}-r\right). \end{cases}$$

The flats of the first set lie entirely in the surface and may be considered as generating the surface, the Cartesian equation of which is obtained by eliminating the parameters ρ_{2j}. The flats of the second set intersect those of the first in points on the surface, hence the parametric equation (89) may be gotten by solving I and II for x_i, y_i, and z.

Since to every sphere of radius $R \neq 0$ correspond two flats which do not intersect, to every surface in S_{n-1} correspond two surfaces in $\bar{\bar{S}}_{n-1}$; these are said to be conjugate to each other.* The conjugate of the surface (84) is obtained by changing the sign of R_1 and R_2 so that we have

$$\frac{a+d}{2}=R_1=\frac{1}{2}\,[2\Sigma\rho_{2i-1}^2+1-k],\quad R_2=-\frac{1}{2}\,[2\Sigma\rho_{2i}^2+1+k],$$

$$\frac{a-d}{2}=-\frac{1-2\Sigma\rho_{2i-1}^2}{2},\quad \frac{a-d}{2}=\frac{1-2\Sigma\rho_{2i}^2}{2},$$

from which we have

$$a=2\Sigma\rho_{2i-1}^2-\frac{k}{2},\quad d=1-\frac{k}{2};\quad a=-2\Sigma\rho_{2i}^2-\frac{k}{2},\quad d=-\frac{k+2}{2},$$

the conjugate surface is therefore generated by either one of the two sets of flats

$$\left.\begin{aligned} x_i&=\left(2\Sigma\rho_{2i-1}^2-\frac{k}{2}\right)y_i+2\rho_{2i-1}, & x_i&=-\left(2\Sigma\rho_{2i}^2+\frac{k}{2}\right)y_i-2i\rho_{2i},\\ z&=2\Sigma\rho_{2i-1}y_i+1-\frac{k}{2}, & z&=2i\Sigma\rho_{2i}y_i-\frac{k+2}{2}.\end{aligned}\right\}\quad(91)$$

Eliminating the ρ's we have the quartic surface

$$\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2-2\sum_1^{\frac{n-2}{2}}x_iy_i-k\Sigma y_i^2+z(z+k)+\frac{k^2-4}{4}=0,\quad (i<k),\quad (84'')$$

which intersects the quadric (84′), *i. e.*, its conjugate, along the null-flats of the first and second sets, that is, flats for which

$$2\Sigma\rho_{2i-1}^2+1-k=0 \text{ and } 2\Sigma\rho_{2i}^2+1+k=0,$$

(equations (87) and (88)).

If the cyclide is of type $r\neq\frac{n-2}{2}$ the equations of the surface conjugate to (90) are

$$\left.\begin{aligned} &\sum_1^r(x_iy_k-x_ky_i)^2-2\sum_1^r x_iy_i-k\sum_1^r y_i^2+z(z+k)+\frac{k^2-4}{4}=0,\\ &\sum_1^r(y_ix_{r+s}-x_ky_{r+s})y_i+zy_{r+s}+\frac{k-2}{2}\,y_{r+s}=0,\quad s=1,2,\ldots,\frac{n-2}{2}-r.\end{aligned}\right\}\quad(90')$$

*Also "reciprocal" in the terminology of S. Lie.

Since the flats of the second set do not lie on the surface (90) the null-flats of the second set will not lie on the intersection of (90) and (90′); only if $r=\frac{n-2}{2}$ will this be true.

§ 3. *Transformation of the Cyclides of the Fourth Order.*

We write the two sets of spheres which generate the cyclide of type $r=\frac{n+2}{2}$ as follows:*

$$\left.\begin{aligned}
&\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-3}X_2+\ldots+2\cot\theta_{n-2}\sin\theta_{\frac{n}{2}}\cdot X_{n-2}\\
&\qquad+\frac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-\frac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2},\\
&\Sigma X_i^2-2\sin\theta_{\frac{n-2}{2}}\ldots\sin\theta_1X_1-\ldots-2\sin\theta_{\frac{n-2}{2}}\cos\theta_{\frac{n-4}{2}}X_{n-3}\\
&\qquad-\frac{2\cos\theta_{\frac{n-2}{2}}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-2\left[1-\frac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{\frac{n-2}{2}}\right].
\end{aligned}\right\}\quad(92)$$

Transforming we get two sets of flats,

$$\text{I(a)}\begin{cases}x_i=ay_i+b_i,\\ z=-\sum_1^{\frac{n-2}{2}}b_iy_i+d,\end{cases}\qquad \text{II(a)}\begin{cases}x_i=\bar a y_i+\bar b_i,\\ z=\sum_1^{\frac{n-2}{2}}\bar b_iy_i+\bar d,\end{cases}$$

where a, b_i, d, $\bar a$, $\bar b_i$, $\bar d$ have the following values:

$$\begin{aligned}
a&=-\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k, &\bar a&=\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}+k,\\
d&=-\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k, &\bar d&=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}+k,\\
b_i&=i\cot\theta_{n-2}\phi_i, &\bar b_i&=-\sin\theta_{\frac{n-2}{2}}\psi_i,
\end{aligned}$$

ϕ_i and ψ_i having the values:

$$\begin{aligned}
\phi_1&=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-4}\cos\theta_{n-3}, &\phi_{i+1}&=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-(i+3)}\sin\theta_{n-(i+2)},\\
\psi_1&=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_1, &\psi_{i+1}&=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_i\cos\theta_{i-1},
\end{aligned}$$
$$\left(i=1,2,\ldots,\frac{n-4}{2}\right).$$

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* These equations differ from (46), only in orientation of the axis $X_1,\ldots,X_{n-2}$; we also have put $a=a_0$ in order to avoid confusing it with the parameter a in I(a) and II(a).

Eliminating the θ's from I (or II) we obtain the quartic surface:

$$(1+a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{\frac{n-2}{2}}(x_i-ky_i)^2-(1+a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1+a_0)(z-k)-1+a_0=0, \quad i<k. \quad (93)$$

The conjugate surface is:

$$(1-a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{\frac{n-2}{2}}(x_i+ky_i)^2-(1-a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1-a_0)(z+k)^2-(1+a_0)=0, \quad (93')$$

generated by the two sets of flats:

$$\text{I(b)}\begin{cases}x= a'y_i+b_i,\\ z=-\Sigma b_iy_i+d',\end{cases} \qquad \text{II(b)}\begin{cases}x_i=\bar{a}'y_i+\bar{b}_i,\\ z=\Sigma\bar{b}_iy_i+\bar{d}',\end{cases}$$

where a', d', $\bar{a}'$, $\bar{d}'$ have the values

$$a'=\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}-k, \qquad \bar{a}'=\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}-k,$$

$$d'=\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}-k, \qquad \bar{d}'=-\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}-k.$$

These two surfaces intersect along the two sets of null-flats whose equations are:

$$x_i=-a_0ky_i+i\sqrt{1+k^2(1-a_0^2)}\phi_i, \qquad x_i=-\frac{k}{a_0}y_i-\frac{\sqrt{a^2-k^2(1-a_0^2)}}{a_0}\psi_i,$$

$$z=-\sum_1^{\frac{n-2}{2}}i\sqrt{1+k^2(1-a_0^2)}\phi_iy_i+ak, \qquad z=-\sum\frac{\sqrt{a_0^2-k^2(1-a_0^2)}}{a_0}\psi_iy_i+\frac{k}{a_0}.$$

These flats are the intersections of the quadric cylinders

$$\Sigma(x_i+a_0ky_i)^2=-[1+k^2(1-a_0^2)], \text{ and } \Sigma\left(x_i+\frac{k}{a_0}y_i\right)^2=\frac{a_0^2+k^2(1-a_0^2)}{a_0^2}$$

with the surface (93) or (93′).

If the cyclide is of type $r<\frac{n+2}{2}$ we write the two sets of generating spheres as follows:

$$\left.\begin{aligned}
&\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_2+\ldots\\
&\quad+2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{2(r-2)}+\frac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2\\
&\quad-\frac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2},\\
&\Sigma X_i^2-2\sin\theta_{n-r}\ldots\sin\theta_1X_1-2\sin\theta_{n-r}\ldots\cos\theta_1X_3-\ldots\\
&\quad-2\sin\theta_{n-r}\ldots\cos\theta_{r-3}X_{2(r-2)-1}-2\sin\theta_{n-r}\ldots\cos\theta_{r-2}X_{2(r-2)+1}\\
&\quad-\ldots-2\sin\theta_{n-r}\cos\theta_{n-r-1}X_{n-3}-\frac{2\cos\theta_{n-r}}{\sqrt{1-a_0^2}}X_{n-1}\\
&\quad=1+k^2-2\left[1-\frac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{n-r}\right].
\end{aligned}\right\} \quad (94)$$

The parameters b_i, c_i, a, d, $\bar{b}_i$, $\bar{c}_i$, $\bar{a}$, $\bar{d}$ of the corresponding flats are therefore:

$$\left.\begin{aligned}
b_1&=i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3},\\
b_2&=i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\sin\theta_{n-3},\ldots,\\
b_{r-2}&=i\cot\theta_{n-2}\sin\theta_{n-r+1},\\
a&=-\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k,\quad c_i=-b_i,\qquad i=1,2,\ldots,r-2,\\
d&=-\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k,\quad b_{r-2+s}=c_{r-2+s}=0,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r,\\
\bar{b}_1&=-\sin\theta_{n-r}\ldots\sin\theta_1,\quad \bar{b}_2=\sin\theta_{n-r}\ldots\cos\theta_1,\ldots,\\
\bar{b}_{r-2}&=-\sin\theta_{n-r}\ldots\cos\theta_{r-3},\quad \bar{c}_i=\bar{b}_i,\qquad i=1,2,\ldots,r-2,\\
\bar{b}_{r-2+s}&=-\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}+i\cos\theta_{r-3+2s}),\\
\bar{c}_{r-2+s}&=-\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}-i\cos\theta_{r-3+2s}),\\
\bar{a}&=\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{n-r}+k,\quad \bar{d}=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{n-r}+k,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}\quad(95)$$

The two sets of flats may then be written:

$$\text{I(c)}\begin{cases}
x_i= & ay_i+b_i,\\
x_{r+s-2}= & ay_{r+s-2},\\
z= & -\sum_1^{r-2} b_iy_i+d,
\end{cases}$$

$$\text{II(c)}\begin{cases}
x_i= & \bar{a}y_i+\bar{b}_i\\
x_{r+s-2}= & \bar{a}y_{r+s-2}+\bar{b}_{r+s-2},\\
z= & \sum_1^{r-2}\bar{b}_iy_i+\Sigma\bar{c}_{r+s-2}y_{r+s-2}+\bar{d},
\end{cases}\quad\left\{\begin{aligned}&i=1,2,\ldots,r-2,\\&s=1,2,\ldots,\frac{n+2}{2}-r\end{aligned}\right\}.$$

Eliminating the parameters θ_i from I(c) we obtain a surface of $\frac{n-6}{2}+r$ dimensions whose equations are

$$\left.\begin{aligned}
&(1+a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{r-2}(x_i-ky_i)^2-(1+a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad+(1+a_0)(z-k)^2-(1-a_0)=0,\\
&(1+a_0)\sum_1^{r-2}(x_{r+s-2}y_i-x_iy_{r+s-2})y_i-(1+a_0)(z-k)y_{r+s-2}\\
&\qquad+(1-a_0)(x_{r+s-2}-ky_{r+s-2})=0,\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}(96a)$$

Its conjugate, which may be obtained by changing the sign of a_0 and k in the above equations, has for equations:

$$\left.\begin{aligned}
&(1-a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{r-2}(x_i+ky_i)^2-(1-a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad+(1-a_0)(z+k)^2-(1+a_0)=0,\\
&(1-a_0)\sum_1^{r-2}(x_{r+s}y_i-x_iy_{r+s-2})y_i-(1-a_0)(z+k)y_{r+s-2}\\
&\qquad\qquad+(1+a_0)(x_{r+s-2}+ky_{r+s-2})=0.
\end{aligned}\right\}(96b)$$

The second set of flats which do not lie on the surface intersect those of the first in a point on the surface; the parametric equations of the surface will therefore be found by solving I(c) and II(c) for x_i, y_i, and z. The null-flats of I(c), which also lie on the surface, form part of the intersection of the surface (96a, b) with its conjugate. The second set of null-flats are common tangent flats of the two conjugate surfaces along the remaining locus of intersection in the finite part of space. At infinity there is a common cone of intersection whose equations are:

$$\left.\begin{array}{l} \sum_1^{r-2}(x_i y_k - x_k y_i)^2 = 0, \quad \sum_1^{r-2}(x_{r+1}y_i - x_i y_{r+1})y_i = 0, \\ x_{r+s-2}y_{r+s'-2} - x_{r+s'-2}y_{r+s-2} = 0, \quad s' \neq s, \quad u = 0. \end{array}\right\} \quad (97)$$

The surface (96a) *is the locus of* ∞^{r-2} *flats which intersect a family of* ∞^{n-r} *flats.* Such a locus we shall call a *flat-regulus* and denote it by the symbol $R^{(r-2)}_{\frac{n-6}{2}+r}$, the lower subscript indicating the dimensions of the surface. If a regulus is a quartic n—2-spread, or is the intersection of two or more such spreads, we shall call it a *quartic regulus;* it may of course happen, as in the case of (96a) and (96b), that one or more, but not all, reduce to cubic spreads, quadrics or even flats. A cubic (quadric) regulus is a cubic (quadric) n—2-spread, or the intersection of two or more such spreads. (96a), (96b) and (90′) are quartic reguli, and (90) a quadric regulus. In 5-space, however, $n=6$, the reguli (96a) and (96b) are cubic, as is also the case in any odd space if $r=3$; the regulus has then only a single infinity of flats.

The flats generating the surfaces belong to certain systems of linear flat-complexes which we shall study in a more comprehensive manner in another paper. We shall state the results obtained in the following theorem which is a generalization of the corresponding one for 3-space given by Lie. It is pertinent to point out here the great generality of the theorem, since in 3-space only one type can exist, namely, $r=1$ for cyclides of the third order, and $r=3$ for those of the fourth:

THEOREM. *The generalized flat-sphere transformation* (3) *carries the Dupin cyclide of the third order and type* $r \overline{\leq} \frac{n-2}{2}$ *in* S_{n-1} (n—1 odd) *into a quadric and quartic regulus of* $\frac{n-2}{2}+r$ *dimensions which are conjugate to each other. If* $r=1$ *the quartic regulus is cubic, and in* 5-*space it reduces to a quadric regulus. If* $r=\frac{n-2}{2}$ *the transform is a quartic* n—2-*spread. The*

same transformation carries the Dupin cyclide of the fourth order and type $r \gtreqless \frac{n+2}{2}$ *into two conjugate quartic reguli of dimensions* $\frac{n+2}{2}+r$, *which become cubic if* $r=3$. *In 5-space they are quadric reguli. If* $r=\frac{n+2}{2}$ *the transform is a quartic* $n-2$*-spread which in 5-space has a quadric locus of double-points at infinity.*

The asymptotic directions proceeding from a point on a regulus $R^{(r-2)}_{\frac{n-6}{2}+r}$ lie in the two flats that intersect at the point. There are $\infty^{\frac{n-4}{2}}$ directions which lie in the flat of the first set, and ∞^{r-3} lying in that of the second set. The asymptotic lines are therefore indeterminate to a less extent than the lines of curvature on a cyclide.* For $n=4$ $r=3$ (ordinary space) the number of directions are finite and equal to 2. The group of $\infty^{\frac{\overline{n+1}\cdot\overline{n+2}}{2}}$ contact-transformations whose characteristic functions are given by (2), carries flats into flats, and it therefore transforms all flat-reguli *inter se*. The group is therefore related to all the spreads $R^{(r)}_{\frac{n-2}{2}+r}$ and $R^{(r-2)}_{\frac{n-6}{2}+r}$ in the same way that the projective group of ∞^{15} transformations are related to all the quadric surfaces in 3-space. We have thus obtained a class of surfaces in odd space which from the standpoint of Sphere-Flat Geometry is the generalization of the quadric surface in 3-space, and which we shall meet with again in the study of flat-complexes.

§ 4. *The Asymptotic Lines on a Sphere in Odd Space.*

We have seen that the asymptotic lines on the transform of cyclides in S_{n-1} are to a certain extent indefinite; these spreads are therefore exceptions to the general rule. Thus, for example, the transforms of homofocal or co-axial quadrics have definite asymptotic lines.

The asymptotic lines on a quadric surface have not been determined except for 3-space. Let the quadric be written

$$\Sigma a_i x^2 + \Sigma b_i y^2 + c z^2 = 1. \tag{98}$$

Transforming by Euler's transformation (79) we have the surface

$$F^2 = \frac{1}{c}\left[1 + c\Sigma\frac{\alpha_i^2}{a_i}\right][1 - \Sigma b_i \beta_i^2], \tag{98'}$$

* This is not true relatively. The regulus is a surface of dimensions $\frac{n-6}{2}+r$ while the corresponding cyclide is of $n-2$ dimensions.

so that the tangential equation of the transform in the α_i, β_i coordinates (5) is known, and the parametric equations of the surface may be derived from it. The surface is not real, and the determination of the lines of curvature is not a simple matter. In the case of a sphere a direct method is successful. Let the sphere be written

$$\Sigma x_i^2+\Sigma y_i^2+z^2=r^2; \tag{99}$$

the system of total differential equations which determine the lines, viz.:

$$dx_i dp_{\frac{n-2}{2}}+dy_i dq_{\frac{n-2}{2}}=0, \quad dq_i dp_{\frac{n-2}{2}}-dp_i dq_{\frac{n-2}{2}}=0, \qquad i=1, 2, \ldots, \frac{n-2}{2}, \tag{100}$$

may be replaced by the following:

$$dx_i dp_i+dy_i dq_i=0, \quad dq_1 dp_i-dp_1 dq_i=0, \qquad i=1, 2, \ldots, \frac{n-2}{2}. \tag{100$'$}$$

We have from (99),

$$p_i=-\frac{x_i}{z}, \quad q_i=-\frac{y_i}{z}.$$

Introducing the values of dp_i and dq_i in (100′), and keeping account of the relations $dx_1dy_i-dy_1dx_i=0$, $\left(i=2, 3, \ldots, \frac{n-2}{2}\right)$, the system becomes

$$\left.\begin{aligned} dz(x_i dx_i+y_i dy_i) &= z(dx_i^2+dy_i^2), \\ dz^2(y_1x_i-x_1y_i) &= dz(y_1dx_i-x_1dy_i+x_idy_1-y_idx_1). \end{aligned}\right\} \tag{101}$$

We shall first suppose that $dz \neq 0$. The second set may then be integrated at once giving $\frac{n-4}{2}$ integrals:

$$y_1x_i-x_1y_i=2c_{i-1}z, \qquad i=2, 3, \ldots, \frac{n-2}{2}. \tag{102}$$

In order to integrate the first set we shall introduce the variables u_i and v_i, putting

$$\left.\begin{aligned} x_i+iy_i=u_i, &\quad x_i=\frac{u_i+v_i}{2}, \\ x_i-iy_i=v_i, &\quad y_i=\frac{u_i-v_i}{2_i}, \end{aligned}\right\} \tag{103}$$

so that (90) and (102) become

$$z^2+\Sigma u_i v_i=r^2, \quad u_1v_i-v_1u_i=2c_{i-1}z, \qquad i=2, 3, \ldots, \frac{n-2}{2}, \tag{104}$$

and the first set (101) is now

$$\frac{dz}{z}=\frac{2du_i dv_i}{v_i du_i+u_i dv_i},$$

which, since $dv_1du_i-du_1dv_i=0$, may be written in the form:

$$\frac{du_i}{dz}=\frac{v_idu_1+u_idv_1}{2zdv_1}.$$

Introducing now a factor of proportionality ρ, we may write this system in the final form:

$$\rho du_i=u_idv_1+v_idu_1,\quad \rho dz=2zdv_1. \tag{105}$$

Differentiating (104) we have,

$$u_1dv_i+v_idu_1-v_1du_i-u_idv_1=2c_{i-1}dz,\quad \Sigma(u_idv_i+v_idu_i)=-2zdz, \tag{106}$$

which, together with (105), constitute a system of n linear and homogeneous equations in $n-1$ unknowns du_i, dv_i, dz. This system will be consistent if, and only if, the determinant of the system vanishes, *i. e.*, if

$$\begin{vmatrix}
v_1-\rho & 0 & 0\ldots\ldots & 0 & u_1 & 0 & 0\ldots\ldots 0\\
v_2 & -\rho & 0\ldots\ldots & 0 & u_2 & 0 & 0\ldots\ldots 0\\
v_3 & 0 & -\rho\ldots\ldots & 0 & u_3 & 0 & 0\ldots\ldots 0\\
\cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots\\
\cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots\\
0 & 0 & 0\ldots\ldots & -\rho & 2z & 0 & 0\ldots\ldots 0\\
v_2 & -v_1 & 0\ldots\ldots & -2c_1 & -u_2 & u_1 & 0\ldots\ldots 0\\
v_3 & 0 & -v_1\ldots\ldots & -2c_2 & -u_3 & 0 & u_1\ldots\ldots 0\\
\cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots\\
\cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots\\
v_1 & v_2 & v_3\ldots\ldots & 2z & u_1 & u_2 & u_3\ldots\ldots u_{\frac{n-2}{2}}
\end{vmatrix}=0. \tag{107}$$

Expanding this determinant we find that it is a quadratic in ρ, $\rho^{\frac{n}{2}-2}$ being a factor. The zero value of the root may be neglected as no solution of the system (105) corresponds to it. The quadratic equation is:

$$\rho^2+\frac{4z(u_1z+\sum\limits_{2}^{\frac{n-2}{2}}c_{i-1}u_i)}{\Sigma u_i^2}\rho+\frac{4z^2[\sum\limits_{1}^{\frac{n-2}{2}}c_{i-1}^2-u_1v_1]}{\Sigma u_i^2}=0. \tag{108}$$

We now solve the equations (104) for v_1 after eliminating $v_2, v_3, \ldots, v_{\frac{n-2}{2}}$ and substitute in (108). Solving we have the two roots

$$\rho=\frac{-2z(u_1z+\sum\limits_{2}^{\frac{n-2}{2}}c_{i-1}u_i)\pm 2z\sqrt{R}}{\Sigma u_i^2},$$

where $R=\sqrt{u_1^2r^2-\sum\limits_1 c_{i-1}^2\Sigma u_i^2+(\sum\limits_2 c_{i-1}u_i)^2}$. Introducing ρ in the system (105),

and eliminating the v's by means of (108), we have the following system of differential equations:

$$\frac{dz}{u_1(r^2+z^2)\mp 2z\sqrt{R}} = \frac{du_1}{u_1\Big(u_1z+\sum\limits_{2}^{\frac{n-2}{2}} c_{i-1}u_i\mp\sqrt{R}\Big)}$$

$$= \frac{du_2}{u_2\Big(u_1z+\sum\limits_{2}^{\frac{n-2}{2}} c_{i-1}u_i\mp\sqrt{R}\Big)-c_1\Sigma u_i^2} = \dots\dots$$

$$= \frac{du_{\frac{n-2}{2}}}{u_{\frac{n-2}{2}}\Big(u_1z+\sum\limits_{2}^{\frac{n-2}{2}} c_{i-1}u_i\mp\sqrt{R}\Big)-c_{\frac{n-4}{2}}\Sigma u_i^2}. \qquad (109)$$

By properly combining we easily find the following algebraic integrals:

$$\frac{u_{i+2}}{c_{i+1}}=\frac{u_2}{c_1}+k_iu_1,\qquad i=1,2,\dots,\frac{n-6}{2}, \qquad (110)$$

the k's being constants of integration. The remaining integrals may now be found as follows: We substitute the values of the u's from (110) in the expression for R, above, and find

$$R=\sqrt{r^2-\Big(1+\sum_1^{\frac{n-6}{2}}k_i^2\Big)\sum_1^{\frac{n-4}{2}}c_i^2+\Big(\sum_1^{\frac{n-6}{2}}k_ic_{i+1}\Big)^2}\,u_1=Au_1;$$

we also have

$$\sum_2^{\frac{n-2}{2}}c_{i-1}u_i=\sum_1^{\frac{n-6}{2}}k_ic_{i+1}u_1+\frac{\Sigma c_i^2}{c_1}u_2=Bu_1+Cu_2,$$

$$\sum_1^{\frac{n-4}{2}}u_i^2 = (1+\Sigma k_i^2)u_1^2+\frac{2\Sigma k_ic_{i+1}}{c_1}u_1u_2+\frac{\Sigma c_i^2}{c_1^2}u_2^2$$

$$=Du_1^2+\frac{2B}{c_1}u_1u_2+\frac{C}{c_1}u_2^2,\qquad r^2-A^2=c_1CD-B^2.$$

We have then to integrate the equations

$$\frac{dz}{u_1(r^2+z^2)\mp 2zAu_1}=\frac{du_1}{u_1(u_1z+Bu_1+Cu_2\mp Au_1)}$$

$$=\frac{du_2}{u_2(u_1z+Bu_1+Cu_2\mp Au_1)-c_1\Big(Du_1^2+\dfrac{2B}{c_1}u_1u_2+\dfrac{C}{c_1}u_2^2\Big)},$$

which may be further simplified by putting $u_2 = vu_1$:

$$\frac{dz}{r^2+z^2\mp 2Az}=\frac{du_1}{u_1(z+B+Cv\mp A)}=-\frac{dv}{c_1D+2Bv+Cv^2}. \qquad (111)$$

These equations may be integrated by quadratures. We find,

$$z=\frac{\alpha(r^2-A^2)-(Cv+B)}{1+\alpha(Cv+B)}\pm A, \quad u_1=\frac{\beta}{1+\alpha(Cv+B)}\,u_2=\frac{\beta v}{1+\alpha(Cv+B)}. \qquad (112)$$

The remaining u's and all the v's are now obtained from the equations (110) and (104). A rather long but not difficult calculation shows that they may be expressed in the form:

$$u_{i+2}=\frac{p_{i+2}+q_{i+2}v}{1+\alpha(Cv+B)}, \quad v_i=\frac{r_i+s_iv}{1+\alpha(Cv+B)}, \qquad i=1,2,\ldots,\frac{n-2}{2}, \qquad (113)$$

where the coefficients p_i, q_i, r_i, s_i depend on the $n-3$ integration constants c_i, k_i, α, β. *From these equations it appears at once that the integral curves are straight lines.* Through any point on the sphere will pass two such lines corresponding to the two roots of (108), or, corresponding to the $+$ and $-$ sign of A in (111).

We shall investigate the remaining asymptotic loci. We assumed $dz \neq 0$. Let $dz=0$. The second set of equations (101) vanish identically, and the equations of the first set are:

$$dz=0, \quad dx_i^2+dy_i^2=0, \qquad i=1,2,\ldots,\frac{n-2}{2},$$

which are satisfied if we put

$$\text{I.}\quad z=C, \quad x_i+iy_i=m_i, \quad \text{or,} \quad \text{II.}\quad z=C, \quad x_i-iy_i=n_i.$$

The first set of flats intersect the sphere in an $\frac{n-4}{2}$ dimensional flat whose equations are:

$$z=C, \quad x_i+iy_i=m_i, \quad C^2+\Sigma m_i(x_i-iy_i)=r^2,$$

and the second set intersects the sphere in a similar flat whose equations are:

$$z=C, \quad x_i-iy_i=n_i, \quad C^2+\Sigma n_i(x_i+iy_i)=r^2.$$

Through any point on the sphere pass two such flats and hence there will be $\infty^{\frac{n-4}{2}}$ asymptotic directions through it and lying in the respective flats. We shall call these loci *asymptotic flats.* The results obtained will now be stated in the following

THEOREM. *There are two sets of asymptotic loci on a sphere in odd-dimensional space. The first set consists of a double family of* ∞^{n-3} *straight*

lines of which through every point on the sphere will pass two. The second set consists of a double family of $\infty^{\frac{n}{2}}$ $\frac{n-4}{2}$*-dimensional flats, the asymptotic flats, and through every point on the sphere will pass two flats and* $\infty^{\frac{n-4}{2}}$ *asymptotic directions lying in each flat.*

For 5-space these flats are straight lines so that we may say:

Through every point on a sphere in 5-space pass four definite asymptotic directions and the four sets of asymptotic lines consist of a four-fold family of ∞^3 *straight lines.*

In 3-space the first set of ∞^1 asymptotic lines remain while the asymptotic flats, being of zero dimensions, become the ∞^2 points of the sphere.

There exist an indefinite number of surfaces whose asymptotic loci are straight lines and flats, namely, all the transforms of the sphere by the transformations of the group (2). Thus, the quadrics

$$\Sigma a_i(x_i^2+y_i^2)+cz^2=r^2$$

belong to this class, since the transformation,

$$x_i=a_i x_i', \quad y_i=a_i y_i', \quad z=cz',$$

carries flats into flats.

The transform of the sphere in $\bar{\bar{S}}_{n-1}$ is a sextic surface in S_{n-1} whose equation is:

$$[r^2(1+X_{n-1}^2)-\Sigma(X_{2i-1}+iX_{2i})^2]\,[1+X_{n-1}^2+\Sigma(X_{2i-1}-iX_{2i})^2]=X_{n-1}^2(1+\sum_1^{n-1}X_i^2)^2,$$

which has the absolute as locus of double points. The surface has four focal sheets if $n \gtreqless 6$. *In 5-space the surface has circular lines of curvature in all four systems;* it therefore presents a striking resemblance to the cyclides in ordinary space.

WEST VIRGINIA UNIVERSITY, *January* 10, 1916.

Irrational Involutions on Algebraic Curves.

By Joseph Vital DePorte.

§ 1. *General Definitions.*

Given a correspondence between the points of two algebraic plane curves, $F(x_1, x_2, x_3) \equiv F(x) = 0$, of genus p, and $f(y) = 0$, of genus π, such that to an arbitrary point P of the first curve correspond b points of the second, and to an arbitrary point P' of the second correspond a points of the first. The group of b points on $f(y) = 0$, corresponding to P in its turn fixes on $F(x) = 0$ b groups of a points each, and P belongs to each of these groups. Similarly, P' on $f(y) = 0$ belongs to a groups, each consisting of b points. The numbers a, b indicating the number of groups in the correspondence to which arbitrary points on $f(y) = 0$ and $F(x) = 0$ belong, are called the *indices* of the (a, b) correspondence, so that the correspondence on $F(x) = 0$ is of index b, on $f(y) = 0$ of index a.

Every (a, b) correspondence can be expressed by means of two equations between the coordinates of corresponding points, if the two curves are irrational. There is usually a finite number of pairs of points not belonging to the correspondence, the coordinates of which satisfy the equations. If the curves are rational the correspondence can always be expressed by means of one equation. If a point x on $F(x) = 0$ is defined in terms of a parameter λ, and a point y on $f(y) = 0$ in terms of μ, then the (a, b) correspondence may be defined by the polynomial $\Sigma \phi_i(\lambda) \cdot \mu^{b-i}$, in which each $\phi_i(\lambda)$ is a polynomial of order a in λ. It will be stated later when this can be done even when the curves are irrational.

If either a or b is one, the coordinates of P or P' can be expressed rationally in terms of the coordinates of P' or P. If both a and b are equal to one, that is if the correspondence is one-to-one, then it is birational, and the coordinates of points on each curve can be expressed rationally in terms of the coordinates of points on the other. In this case the curves $F(x) = 0$, $f(y) = 0$ are said to be birationally equivalent.

A correspondence of index 1, for instance $(a, 1)$, is called an *involution* of order a. The genus of the curve, the points of which are in $(1, a)$ corre-

spondence with the points of the given curve, is called the *genus of the involution*. If the genus of the involution is zero it is said to be rational; otherwise, irrational.

Every rational involution is a linear series and vice versa.

If a curve possesses two rational or irrational involutions, γ'_m, $\mu'_n (m \leq n)$ such that every point of a group of the first belongs to a distinct group of the second, in which case the groups of μ'_n, taken m at a time are conjugate under γ'_m, we say that γ'_m is compounded with μ'_n.

Given a curve possessing a γ'_t. If an arbitrary point determines not 1, but k distinct groups of the involution, the involution is said to be multiple.

§ 2. *Branch Points and Coincidences.*

If in a general (a, b) correspondence two or more points of a group of points corresponding to a given point P of a curve coincide, the point P is called a branch-point. The numbers of simple branch-points y and y' (that is those for which only two of the corresponding points coincide) in an (a, b) correspondence between curves of genera p and π are connected by the following relation, due to Zeuthen:*

$$y-y'=2a(\pi-1)-2b(p-1). \tag{1}$$

If the correspondence is involutorial, for example $(a, 1)$, then $y=0$, and we have the theorem:

The number of double points of an involution of order a, genus π, on a curve of genus p is

$$y'=2(p-1)-2a(\pi-1). \tag{2}$$

If the correspondence between the curves is $(1, 1)$, then (1) furnishes a direct proof of Riemann's theorem concerning the equality of the genera of curves that are birationally equivalent. For, if $a=b=1$, then $y=y'=0$, and we have $2(\pi-1)=2(p-1)$, hence $p=\pi$.

From Zeuthen's formula also follows † that there do not exist involutions of genus $p (p>1)$ on curves of the same genus.

For, setting $\pi=p$ in (2),

$$y'=2(p-1)-2a(p-1).$$

But $y'\geq 0$, $\therefore\ p-1\geq a(p-1)$, $\therefore\ a=1$.

* H. G. Zeuthen, "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes." *Math. Annalen*, Vol. III (1870), pp. 150–156.

† Weber, *Jour. für Math.*, Vol. LXXVI (1876), p. 345.

§ 3. *Valence of Involutions.*

If the curves in (a, b) correspondence coincide, then we have an (a, b) correspondence between points of one curve. If as a point P (P') moves along the curve the b (a) corresponding points, together with the point P (P') counted γ times, $(\gamma \gtreqless 0)$, moves in a linear series, the correspondence is said to be of valence γ. In other words, given P and a group of b points corresponding to it. Let P go into P_1, b into b_1, then if the correspondence is of valence γ,

$$\gamma P + b \equiv \gamma P_1 + b_1 \quad (\gamma \gtreqless 0),$$

or the groups $\gamma P + b$, $\gamma P_1 + b_1$ belong to the same linear series of order $\gamma + b$. If γ is negative, (3) may not have a geometric meaning, but by transposing the term containing γ we get

$$\gamma P_1 + b \equiv \gamma P + b_1 \quad (\gamma > 0),$$

which can be interpreted to mean that $\gamma P_1 + b$, $\gamma P + b_1$ belong to the same linear series of order $\gamma + b$.

The number of coincidences z in a valence correspondence as given by the Cayley-Brill-Hurwitz formula is

$$z = a + b + 2p\gamma. \tag{3}$$

§ 4. *Notation.*

We shall represent involutions by small letters, with subscripts indicating the order and genus, and superscript 1 indicating the dimension, using letters of the Latin alphabet for rational involutions (linear series), and of the Greek alphabet for irrational involutions. Thus, $\gamma'_{a,\pi}$ reads: "An irrational involution of order a genus π." g'_a—"a rational involution, or linear series, of order a." Thus $g'_a \equiv \gamma'_{a,0}$. Capital letters are used to indicate individual groups: Γ_a, G_a.

§ 5. *Application to Cubic Curves.*

Every cubic has a single infinity of rational involutions of order 2. In fact, consider a pencil of lines with vertex at an arbitrary point S of the cubic. Each line of the pencil cuts the cubic in two points, each of which uniquely determines the other, since it fixes a line of the pencil. We have then an involution of pairs of points, and since the vertex S is arbitrary, we can construct a single infinity of such central involutions. The involutions are rational, for if we take the point S as the point $(0, 0, 1)$, for example, the equation of any line of the pencil is of the form $x + \lambda y = 0$. The coordinates of a point P on the cubic fix the value of the parameter λ, and we can express

the coordinates of the point P', conjugate to P in the involution rationally in terms of the parameter.

In general, if the lines joining pairs of points of a simple involution of order 2 pass through a point or envelope a rational curve the involution is rational. We saw the truth of the first statement. In the second case the points of the curve enveloped by the lines are in (1, 1) correspondence with the groups of the involution, since to every point of the curve corresponds a line determining a group of the involution. The genus of the curve is, by definition, the genus of the involution. If the curve is rational, so is the involution.

We can construct on a cubic of genus 1 involutions of order 2 which are not rational by taking the product of two central involutions. Let us take a point $S(s)$, (we shall thus indicate the parameter, in terms of elliptic functions of which the coordinates of a point on the cubic can be expressed rationally) as center and project from it an arbitrary point $P(p)$ on the curve into $P_1(p_1)$. Project then $P_1(p_1)$ from another center $S_1(s_1)$ into $P_2(p_2)$. Repeat the process by projecting P from S_1 into $P'(p')$, and P' from S into $P''(p'')$. If $P_2=P''$, we have an involution of order 2 of which P and P_2 are a pair.

The necessary and sufficient condition that $P_2\equiv P''$ is that the parameters of S and S_1 differ by half a period.

The sum of the parameters of three collinear points on an elliptic curve is congruent to zero.

The points S, P, P_1 are collinear, hence

$$s+p+p_1\equiv 0, \quad \text{or} \quad p_1\equiv -(s+p).$$

The points P_1, S_1, P_2 are collinear, hence

$$-(s+p)+s_1+p_2\equiv 0, \quad \text{or} \quad p_2\equiv s+p-s_1.$$

Also the points P, S_1, P' lie on a straight line, hence

$$s_1+p+p'\equiv 0, \quad \text{or} \quad p'\equiv -(s_1+p).$$

And the points P', S, P'' are collinear, hence

$$-(s_1+p)+s+p''\equiv 0, \quad \text{or} \quad p''=s_1+p-s.$$

$$\text{If } P''\equiv P_2, \; p''\equiv p_2, \; \text{or} \; s+p-s_1\equiv s_1+p-s,$$

$$\therefore 2s\equiv 2s_1 \pmod{\omega, \omega'},$$

hence the parameter of S_1 differs from that of S by half a period. Conversely, if s and s_1 differ by half a period, $P''=P_2$. Since the parameter of P does not enter in the last equation the statement is true for any point on the cubic.

If we draw the tangent to the cubic at S it will cut the curve again in one point $O(o)$. From O we can draw three tangents to the cubic, different from the tangent at O and OS. Let the points where the three tangents touch the curve be $S_1(s_1)$, $S_2(s_2)$ and $S_3(s_3)$. We have then the following four relations between the parameters of the points O, S, S_1, S_2, S_3,

$$2s+o\equiv 0, \quad 2s_1+o\equiv 0, \quad 2s_2+o\equiv 0, \quad 2s_3+o\equiv 0.$$

Eliminating o we find that the parameters of S_1, S_2, S_3 differ from the parameter of S by half a period.

It can easily be seen that the converse is also true, namely, if the parameters of two points differ by half a period, the tangents to the cubic at these points intersect in a point on the curve.

Collecting the above results we can state that the necessary and sufficient condition that the product of two central involutions on a cubic of genus 1 is an involution of order 2 is that the tangents to the cubic at the centers of the involutions intersect in a point on the cubic.

Associated with any point $S(s)$ on the cubic there are three points $S_1\left(s+\frac{\omega}{2}\right)$, $S_2\left(s+\frac{\omega'}{2}\right)$, $S\left(s+\frac{\omega+\omega'}{2}\right)$ such that

$$SS_1\equiv S_1S, \quad SS_2\equiv S_2S, \quad SS_3\equiv S_3S.$$

It is important to notice that we get the same three involutions, no matter where S is taken. In fact, let $SS_1\equiv S_1S$, also $C(t)C_1(t_1)\equiv C_1(t_1)C(t)$. Under S a point $P(p)$ goes into $P_1(p_1)$, and P_1 under S_1 goes into $P_2(p_2)$, so that

$$s+p+p_1\equiv 0, \quad s_1+p_1+p_2\equiv 0.$$

Eliminating p_1, we have

$$p+(s-s_1)+p_2\equiv 0.$$

If the point P goes under C into $P'(p')$ and P' under C_1 goes into $P''(p'')$, we have:

$$t+p+p'\equiv 0, \quad t_1+p'+p''\equiv 0.$$

Hence, eliminating p',

$$p+(t-t_1)+p''\equiv 0.$$

Since SS_1, and CC_1 are, by hypothesis, involutions,

$$s-s_1\equiv t-t_1.$$

Therefore $p_2\equiv p''$, and the point P has the same conjugate in both involutions. The involutions are, therefore, identical.

The lines joining pairs of corresponding points envelope a curve of genus 1—the genus of elliptic functions in terms of which the equations of the lines can be expressed rationally. Hence, if the product of two central involutions on a cubic curve of genus 1 is an involution of order 2 its genus is 1.

§ 6. *General Theorems.*

Rational involutions have been studied in detail.* Comparatively little has been done in the field of irrational involutions. Castelnuovo † derived the following theorem: *A group of a* $\gamma'_{a,\pi}$ *on a curve of genus p has but* $a-1$ *conditions to belong to a group of a* g^r_n, *if* $n-r<p-a\pi$. For $\pi=0$ and $g^r_n\equiv g^{p-1}_{2p-2}$ the theorem reduces to the Riemann-Roch theorem.

Amodeo ‡ derived from the Zeuthen and Cayley-Brill-Hurwitz formulas a number of theorems on the range of possible involutions on curves of given genus. In so far as the theorems refer to irrational involutions on curves of genus greater than 1, and of general moduli they are of no value, since, as will be pointed out later, such involutions do not exist.

In a later paper by Castelnuovo § appears the important theorem:

The necessary and sufficient condition in order that a simply infinite series γ'_a *of order a, and of index b, on a curve* $F(x)=0$ *of order n, genus p, belongs to a linear series* g'_a *of the same order, is that the series shall possess* $2b\ (a+p-1)$ *double points.*

The proof, in brief, is as follows:

Construct on $F(x)=0$ a non-special linear series (that is a series g^r_m, where $m-r=p$. It can be cut out by adjoints of order greater than $n-3$), of dimension $a-1$. The difference between the order and dimension of a non-special series being p, its order will be $a-1+p$. The series can always be selected in such a way that no given complete group of a points of γ'_a belongs to a group of the new series. For, we can choose $a-1$ points of a group of γ'_a, and add to them p points taken arbitrarily, but so as not to contain the a-th point of the group. We will have then a group of the series g^{a-1}_{a-1+p}. But this one group will fix the series. Since every group will contain the same p points and $a-1$ other points, we will have constructed a series of the kind desired.

Applying the Segre formula ‖ we find that the number of groups of a points that the linear series g^{a-1}_{a-1+p} and γ'_a have in common is

$$z=b(a+p-1)-1/2d,$$

* For the general theory on linear series see Clebsch-Lindemann, "Vorlesungen über Geometrie" (1876); Severi, "Lezioni di Geometria Algebrica" (1910, lith.). A list of references to the recent literature is given in Doehlemann's "Geometrische Transformationen," zweiter Teil (1908), p. 174.

† G. Castelnuovo, "Alcuni osservazioni sopra le serie irrazionali di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 4, Vol. VII² (1891), pp. 294-299.

‡ F. Amodeo, "Contribuzione alla teoria delle serie irrazionali involutorie giacenti sulle varieta algebriche ad una dimensione," *Ann. di Mat.*, s. 2, Vol. XX (1892), pp. 227-235.

§ G. Castelnuovo, "Sulle serie algebriche di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 5, Vol. XV (1906), pp. 337-344.

‖ C. Segre, "Sulle varieta algebriche di una serie semplicemente infinita di spazi," *Rom. Acc. Lincei Rend.*, (4), Vol. III² (1887), pp. 149-153.

where d is the number of double points of γ'_a. It follows that

$$d=2b(a+p-1)-2z.$$

Since $z\geq 0$,

$$d\leq 2b(a+p-1).$$

If d has the value given by the equality sign, z will be equal to zero, and the linear series g^{a-1}_{a-1+p} will not contain any groups of γ'_a. But if a series g^{a-1}_{a-1+p} be constructed to contain in one of its groups a group of γ'_a (which is possible in ∞^{p-1} ways), then the series will contain γ'_a entirely.

Now construct another linear series, g^a_{a+p}, a group of which is to be made up of a group of γ'_a, Γ_a, and p arbitrary points: $c_1, \ldots, c_p$. Then the series g^{a-1}_{a-1+p}, residual with respect to c_i will contain Γ_a, hence all other groups of γ'_a. In other words the groups residual to the groups of γ'_a with respect to g^a_{a+p} will all pass through c_i, where $i=1, 2, \ldots, p$. Hence we have a group G_p residual to any Γ with respect to g^a_{a+p}. In consequence, γ'_a belongs to linear series g_a, which is residual to G_p with respect to g^a_{a+p}.

Conversely, if γ'_a belongs to a linear series g_a, then a linear series g^{a-1}_{a-1+p} which contain one group Γ will contain all. If a series g^{a-1}_{a-1+p} be constructed so as not to contain in any of its groups a given group of γ'_a, it will not contain any group of γ'_a. Hence z will be equal to zero, and $d=2b(a+p-1)$.

Stated in other words, the theorem given as the necessary and sufficient condition that an algebraic correspondence (a, b) between two curves of genera p, π can be expressed by means of a single equation (rational in the coordinates of corresponding points) is that the number of branch-points on one curve is $2b(a+p-1)$, and on the other $2a(b+\pi-1)$, and conversely.

If the correspondence is involutorial, for instance $(a, 1)$, then the series γ'_a is a rational series g'_a, if it has $2(a+p-1)$ double points. This can be seen also from Zeuthen's formula, for, if in (2) we set $y'=2(a+p-1)$, we get $\pi=0$; and, conversely, if $\pi=0$, $y'=2(a+p-1)$. Thus, a central involution on a non-singular cubic, $(a=2, p=1)$ has four double points and is rational, while the involution obtained by taking the product of two central involutions has no double points and is of genus 1.

§ 7. *Statement of the Problem.*

The purpose of this paper is two-fold:

I. To find the range of all possible involutions on curves of given characteristics.

II. Given an involution, to determine the restrictions on a curve of given genus that it may possess this involution.

In order to ascertain the genus of an involution of order a with a given number of coincidences on a curve $F(x)=0$ of genus p, it will suffice to find any curve $f(y)=0$ in $(1, a)$ correspondence with $F(x)=0$. If the genus of $f(y)=0$ is π, the involution will be of genus π. The curve $F(x)=0$ can not have another involution of the same order and the same number of coincidences but different genus. For, suppose it has beside $\gamma'_{a,\pi}$ also $\gamma'_{a,\pi'}$, that is, let there be a curve $\phi(y')=0$ of genus π' also in $(1, a)$ correspondence with $F(x)=0$. Then,

$$2(p-1)-2a(\pi-1)=2(p-1)-2a(\pi'-1), \therefore \pi=\pi'.$$

We shall arrange involutions according to the genus of the curve $F(x)=0$. Every curve of genus p, $p=3\pi+(0, 1, 2)$, if not hyperelliptic, can be reduced to a curve of order not greater than $2\pi+2$, $2\pi+3$, $2\pi+4$ with $2\pi(\pi-1)$, $2\pi^2$, $2\pi^2+2\pi+1$ double points, and we shall, consequently, consider for every genus curves of the lowest order, the so-called normal curves. Hyperelliptic curves will be treated separately.

§ 8. *Involutions on Rational Curves.* $p=0$.

Setting $p=0$ in (2) we get:

$$y'=-2-2a(\pi-1)\text{; since } y'\geq 0,\ -1-a(\pi-1)\geq 0,$$
$$\therefore a(1-\pi)\geq 1. \text{ But } a>0, \therefore \pi=0.$$

Hence: *Irrational involutions do not exist on rational curves.**

An interesting application of this theorem is found by studying the asymptotic lines of certain ruled surfaces. Given a ruled surface of order $m+n$, having one m-fold directrix line and an n-fold directrix line. The genus of a plane section is

$$\frac{(m+n-1)(m+n-2)}{2}-\frac{m(m-1)}{2}-\frac{n(n-1)}{2}-\Psi=(m-1)(n-1)-\Psi,$$

where Ψ is the number of double generators.

The asymptotic lines are all algebraic and each belongs to a linear complex containing the congruence defined by the directrices. Every generator meets each asymptotic line in two points, and a plane section in one point. Hence, by our theorem we have examples of curves belonging to a linear complex that are not rational.†

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* For a different proof of the same theorem see Lüroth, "Beweis eines Satzes über rationale Curven," *Math. Annalen*, Vol. IX (1876), p. 163.

† See C. P. Steinmetz, "On the Curves Which are Self-Reciprocal in a Linear Null System, and Their Configurations in Space," American Journal of Mathematics, Vol. XIV (1892), pp. 161–186; V. Snyder, "Asymptotic Lines on Ruled Surfaces Having Two Rectilinear Directrices," *Bulletin American Mathematical Society*, Vol. V (1899), pp. 343–353, and "Twisted Curves Whose Tangents Belong to a Linear Complex," American Journal of Mathematics, Vol. XXIX (1907), pp. 279–288. Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces" (1906), pp. 204–220.

§ 9. *Hyperelliptic Curves.*

Curves of genera 1 and 2 belong to the class of hyperelliptic curves, and it will be appropriate to take up at this point the study of involutions on hyperelliptic curves generally. This was done by Torelli,* and we shall reproduce his main results.

Irrational involutions on hyperelliptic curves are hyperelliptic. That is, if a curve $f(y)=0$ is in $(1, a)$ correspondence with a hyperelliptic curve $F(x)=0$ it is itself hyperelliptic. For, as is known, the $\gamma'_{a,\pi}$ on $F(x)=0$ is compounded with the g'_2 of the curve. The pairs of groups of $\gamma'_{a,\pi}$ conjugate under g'_2 form a rational involution on the double line of $F(x)=0$. The groups of the involution are in $(1, 2)$ correspondence with the points of $f(y)=0$. The latter, then, has a g'_2, and is hyperelliptic.

In particular, curves of genera 1 or 2 can not have irrational involutions other than of genus 1. For, setting $p=1$ in (2), we have:

$$y'=-2a\pi+2a,$$

$$\pi=\quad 1-\frac{y'}{2a}, \therefore \pi=0 \text{ or } 1.$$

If $p=2$,

$$y'=2-2a\pi+2a,$$

$$\pi=1-\frac{y'-2}{2a}, \therefore \pi=0 \text{ or } 1.$$

Given a hyperelliptic curve of order $2\pi+2$, genus π,

$$y_1^2=\prod_{i=1}^{2\pi+2}(x_1-\alpha_i), \quad \alpha_i \neq \alpha_k. \tag{4}$$

Applying the transformation

$$x_1=\frac{f(x)}{\phi(x)}, \quad y_1=\frac{y}{[\phi(x)]^{\pi+1}}, \tag{5}$$

where $f(x)$, $\phi(x)$ are relatively prime polynomials of degree a we get the hyperelliptic curve

$$y=\prod_{i=1}^{2\pi+2}[f(x)-\alpha_i\phi(x)]=R(x), \tag{6}$$

where $R(x)$ is a polynomial of order $a(2\pi+2)$. The hyperelliptic curves (4) and (6) are in $(1, a)$ correspondence. The curve (6) has an involution (hyperelliptic) of order a, genus π, a $\gamma^1_{a,\pi}$ which is represented on the x-axis ($y=0$) on which (6) is mapped doubly by the rational involution,

$$f(x)-x_1\phi(x)=0.$$

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* R. Torelli, "Sulle involuzioni irrazionali nelle curve iperellitiche," *Palermo Rend.*, Vol. XIX (1905), pp. 297-304.

To determine the genus of the curve (6) we notice that the factors of $R(x)$ have no common roots. All or some of them may have multiple roots, of even or odd multiplicity. In that case

$$R(x) = [S(x)]\,T(x), \tag{6'}$$

where $T(x)$ is a polynomial of order $2p+1$ or $2p+2$, with simple roots only. Transforming (6) birationally by means of

$$x=x', \quad y=y'S(x'),$$

we obtain

$$y'^2 = T(x) = \prod_{i=1}^{\substack{2p+1\\ \text{or } 2p+2}} (x-b_i), \quad b_i \neq b_k. \tag{7}$$

(7) is of genus p. (6) is in (1, 1) correspondence with it. Hence the genus of (6) is also p.

The coincidences of the curve (6) (which are the double points of g_2') form $2\pi+2$ groups of a rational involution I_a on the x-axis which are made up of the roots of (7) each counted an odd number of times (≥ 1), and the roots of $S(x)$, different from the roots of $T(x)$ (6′), each counted an even number of times (≥ 2). We have at once the following theorem:

The necessary and sufficient condition in order that a hyperelliptic curve $F(x)=0$ of genus p contain an irrational (hyperelliptic) involution of order a, and genus π is that of the $2p+2$ coincidences on the line on which the curve is mapped doubly, each counted an odd number of times and, if necessary, with other points, each counted an even number of times it shall be possible to form $2\pi+2$ groups of a rational involution of order a.

The condition is sufficient. For, let the x-axis on which the hyperelliptic curve $F(x)=0$ is mapped doubly, contain an involution I_a' satisfying the given condition. Then if $f(x)-\alpha_1\phi(x)=0$, $f(x)-\alpha_2\phi(x)=0 \ldots . f(x)-\alpha_{2\pi+2}\phi(x)=0$ are the $2\pi+2$ groups, the curve $y^2 = \prod_{i=1}^{2\pi+2} [f(x)-\alpha_i\phi(x)]$ has the same group of coincidences as $F(x)=0$, and is rationally equivalent to it. Since the former curve is in $(a, 1)$ correspondence with $y^2 = \prod_{i=1}^{2\pi+2}(x-\alpha_i)$, $F(x)=0$ will also be in $(a, 1)$ correspondence with it, and will have a $\gamma'_{a,\pi}$.

Conversely, if $F(x)=0$ has a $\gamma_{a,\pi}$, it can be put in birational correspondence with $y^2 = \prod_{i=1}^{2\pi+2}[f(x)-\alpha_i\phi(x)]$ by a proper choice of the constants α_i, and the polynomials $f(x)$ and $\phi(x)$. The coincidences of the two curves on the x-axis will then be projective.*

* Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito," *Ann. di Mat.*, serie 2, Vol. XXII (1894), § 67, note.

But this is the condition stated in the theorem.

The particular case of cyclic irrational involutions on hyperelliptic curves has been studied by Wiman *

§ 10. *Non-Hyperelliptic Curves.*

Non-hyperelliptic curves of general moduli have only valence-correspondences, † hence they do not have irrational involutions. In order for a curve to possess an involution the moduli of the curve must be specialized. We shall commence the study of involutions on curves with specialized moduli with those of the lower order, namely, 2.

§ 11. *Irrational Involutions of Order* 2.

An irrational involution of order 2 on a given curve associates the points of the curve in pairs, so that to a point P corresponds a unique point P', which is the conjugate of P in the involution. To the point P' corresponds the point P. Thus the involution defines a birational transformation of period 2 of the curve into itself. In other words, if a curve possesses a γ_2^1 it must remain invariant under a birational transformation of period 2.

We shall consider first irrational involutions of period 2 on curves which remain invariant under the simplest of birational transformations—linear transformations.

§ 12. $p=3$.

The normal form of a non-hyperelliptic curve of genus 3 is a non-singular quartic. Since under any birational transformation that leaves the curve invariant the system of adjoints of order $n-3$, that is straight lines, goes over into a system of adjoints of the same order, the transformation is linear.

Consider a non-singular quartic invariant under the linear transformation:

$$L\equiv\begin{pmatrix} x_1 & x_2 & x_3 \\ -x_1 & x_2 & x_3 \end{pmatrix}\ddagger.$$

Its equation will be of the form

$$x_1^4+x_1^2\phi_2(x_2,\,x_3)+\phi_4(x_2,\,x_3)=0, \tag{8}$$

where the ϕ_i's are homogeneous polynomials in x_2 and x_3 of degree i.

* A. Wiman, "Über die hyperelliptischen Curven und diejenigen vom Geschlecht $p=3$ welche eindeutige Transformationen in sich zulassen," *Bihang till K. Svenska Vet. Akad. Handlingar*, Vol. XXI (1895).

† A. Hurwitz, Über algebraische Correspondenzen und das verallgemeinerte Correspondenzprincip," *Math. Annalen*, Vol. XXVIII (1887), p. 560.

‡ Every linear transformation in the plane of period 2 can be put into this form. A. Hurwitz, 'Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen," *Math. Annalen*, Vol. XXXII (1888), p. 290.

A straight line joining a pair of points P and P' conjugate in the involution will cut the quartic again in two points Q and Q'. Since the points P and P' interchange under the transformation which leaves the quartic invariant, the line PP' will go into itself. Hence, Q, Q' also interchange, in other words, Q and Q' are a pair of the involution. The point P determines not only its conjugate P', but also another pair of the involution. Since P is arbitrary, every point on the quartic determines two pairs of the involution, or

If a curve of genus 3 *has a* γ'_2, *the involution is multiple.*

Since we assume that C_4 is not hyperelliptic, it can not have a g'_2. The lines joining pairs of points of the involution belong to a pencil with vertex at the center of homology (1, 0, 0) not on the curve.

To find the genus of the involution we need only construct a curve in (1, 2) correspondence with (8). By means of the transformation

$$T \equiv \begin{pmatrix} x_1 = \sqrt{y_1 y_3}\,, \\ x_2 = y_2\,, \\ x_3 = y_3\,, \end{pmatrix}$$

(8) goes over into the quartic

$$y_1^2 y_3^2 + y_1 y_3 \phi_2(y_2, y_3) + \phi_4(y_2, y_3) = 0, \tag{8'}$$

which is in (1, 2) correspondence with (8). (8′) has a tacnode at the point (1, 0, 0) and no other multiple points; it is, therefore, of genus 1.

An involution of order 2 *on a non-hyperelliptic curve of genus* 3 *is of genus* 1.

We can make use of Zeuthen's formula (2) to verify the result. $\gamma'_{2,1}$ on the quartic has four coincidences—the points of intersection of the line $x_1 = 0$ with the curve. If in (2) we put $y' = 4$, $p = 3$, $a = 2$, we have

$$4 = 2(3-1) - 2 \cdot 2(\pi - 1), \text{ or } \pi = 1.$$

§ 13. $p = 4$.

Normal form of a non-hyperelliptic curve of genus 4 is a quintic with two double points. The equation of a quintic with two double points at (0, 1, 0) and (0, 0, 1) invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + x_1^2 \phi_3(x_2, x_3) + a x_2^3 x_3^2 + b x_2^2 x_3^3 = 0. \tag{9}$$

The center (1, 0, 0) of the homology is a simple point on the quintic. A line joining a pair of points in the involution passes through (1, 0, 0) and cuts the curve again in two points Q and Q'. By the same method as in the previous case we may therefore state the theorem:

If a curve of genus 4 *has a* γ_2', *and is invariant under a linear transformation, the involution is multiple.*

Under T (9) goes over into the quartic

$$y_1^2 y_3 \phi_1(y_2, y_3) + y_1 \phi_3(y_2, y_3) + a y_2^3 y_3 + b y_2^2 y_3^2 = 0. \qquad (9')$$

(9′) is of genus 2, since it has one double point at (1, 0, 0).

An involution of order 2 *on a curve of genus* 4 *is of genus* 2.

The line $x_1=0$ cuts the quintic in one point besides the double points (0, 1, 0) and (0, 0, 1). The center (1, 0, 0) is also a coincident point. Hence the involution has two coincidences. Setting in (2) $y'=2$, $p=4$, $a=2$, we have,

$$2=2(4-1)-2\cdot 2(\pi-1), \text{ or } \pi=2.$$

§ 14. $p=5$.

Curves of genus 5, which do not possess a g_3' can be reduced to a sextic with five double points. A curve of genus 5 having a g_3' can be reduced to a quintic with one double point.

a. If a sextic remains invariant under L and does not pass through the center of homology (1, 0, 0), its equation is of the form

$$x_1^6 + x_1^4 \phi_2(x_2, x_3) + \ldots = 0.$$

If the center of homology is on the sextic it is a double point, for the equation of the curve is then of the form

$$x_1^4 \phi_2(x_2, x_3) + \ldots = 0.$$

In either case, since lines passing through the center of homology and joining pairs of points conjugate in the involution, go over into themselves under the transformation which interchanges the points, every point on the sextic determines in the first case three, and in the second case two groups of the involution.

b. The equation of a quintic of genus 5 invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + \ldots = 0,$$

the double point being on the axis of homology $x_1=0$. The center of homology (1, 0, 0) is on the curve. A line joining a pair of points conjugate in the involution cuts the curve in another pair of points, which also belongs to the involution. We may conclude, then, that *if a curve of genus* 5 *has a* γ_2', *and is invariant under a linear transformation, the involution is multiple.*

§ 15. *Involution of Order* 2 *on Curves of Any Genus.*

In general, *if a curve of any genus greater than* 1, *not hyperelliptic, and of any order, has a* γ_2' *and is invariant under a linear transformation, the invo-*

lution is multiple. For, suppose the order of the normal curve to which the given curve can be reduced by birational transformations is n. If n is even, $n=2m$, the normal curve either does not pass through the center of homology $(1, 0, 0)$, or has at the center a singular point of even multiplicity. The equation of the curve is of the form

$$x_1^{2(m-k)}\phi_{2k}(x_2, x_3)+\ldots\ldots=0,$$

where $k\geqq 0$. Every line through the center of homology cuts the curve in $2(m-k)$ points which interchange in pairs under L. An arbitrary point on the curve thus determines $m-k$ groups of the involution.

If n is odd, $n=2m+1$, the equation of the normal curve is of the form

$$x^{2(m-r)}\phi_{2r+1}(x_2, x_3)+\ldots\ldots=0,$$

where $r\geqq 0$. The center of homology is on the curve. Every line through it cuts the curve in $2(m-r)$ points. An arbitrary point on the curve determines $m-r$ groups of the involution.

§ 16. *Equations of Transformation.*

In order to determine the genera of involutions of order 2 on given curves, it is convenient to view the transformation which carries the given curve into a curve in (1, 2) correspondence with it geometrically. If we consider a new plane (y_1, y_2, y_3) such that to any point (x_1, x_2, x_3) in A corresponds one point (y_1, y_2, y_3) in A', but to any point (y_1, y_2, y_3) in A' correspond two points in A, we may write

$$T^{-1}\equiv\begin{pmatrix} x_1^2=y_1 \\ x_2x_3=y_2 \\ x_3^2=y_3\end{pmatrix}, \quad\text{and}\quad T\equiv\begin{pmatrix} x_1=\sqrt{y_1y_3} \\ x_2=y_2 \\ x_3=y_3\end{pmatrix}.$$

T sets up a (2, 1) correspondence between the plane A of the given curve and the plane A' of the curve in (1, 2) correspondence with it. The fundamental elements in the A-plane are one fundamental point, (0, 1, 0), and one fundamental line, $x_3=0$; in the A'-plane there are also one fundamental point, (1, 0, 0), and one fundamental line, $y_3=0$. To the system of lines in the A-plane, $x_1+ax_2+bx_3=0$ corresponds in the A'-plane a system of conics $y_1y_3+(ay_2+by_3)^2=0$ passing through the fundamental point (1, 0, 0) and tangent to the fundamental line $y_3=0$. The basis points of the system of conics are the points of intersection of the conics $y_1y_3+ay_2^2$ and $y_3(y_1+by_2)=0$. The conics of the net have three-point contact at (1, 0, 0), hence one free point of intersection which corresponds to a point of intersection of two lines in the A-plane.

To the system of lines in the A'-plane, $y_1+ay_2+by_3=0$, corresponds in the A-plane the system of conics $x_1^2+ax_2x_3+bx_3^2=0$. The conic $x_1^2+ax_2x_3=0$ and the line-pair $x_3^2=0$ have two fixed points of intersection at the fundamental point (0, 1, 0); the system of conics in the A-plane has therefore two free points of intersection, which correspond to a point of intersection of two lines in the A'-plane.

To a curve of order n (not passing through the fundamental point) generated in the A-plane by a point-pair P, P_1, corresponds in the A'-plane a curve of order n counted twice. If the curve C_n in the A-plane passes k times through the fundamental point (0, 1, 0), its image is a curve of order $n-\frac{k}{2}$ counted twice, together with the fundamental line $y_3=0$ counted k times.*

If C_n cuts the fundamental line $x_3=0$ in n distinct points, its image passes $\frac{n}{2}$ times through the fundamental point (1, 0, 0). When two or more points of intersection coincide, that is if C_n has a multiple point on $x_3=0$, a corresponding number of tangents to the image curve at the point (1, 0, 0) coincide.

If C_n does not pass through the center of homology (1, 0, 0) its equation is of the form

$$x_1^{2m}+x_1^{2(m-1)}\phi_2(x_2, x_3)+\ldots\ldots=0,$$

$n=2m$. If n is odd, $n=2m+1$, the center of homology is on C_n. Under T C_n goes over into

$$y_1^m y_3^m+y_1^{m-1}y_3^{m-1}\phi_2(y_2, y_3)+\ldots\ldots=0.$$

Hence, if a curve of order $2m$ in the A-plane does not pass through the point (1, 0, 0), its image in the A'-plane has m branches touching each other at the point (1, 0, 0).

The center of homology lies on C_n if n is odd, in which case C_n passes through it an odd number of times (≥ 1), or, if n is even, when the center of homology is a singular point on the curve of even multiplicity (≥ 2).

If C_n has a singular point of order k on the axis of homology $x_1=0$, we can without loss of generality take the point as the point (0, 0, 1). If k is even, $k=2s$, the equation of C_n is of the form

$$x_3^{n-2s}\phi_s(x_1^2, x_2^2)+\ldots\ldots=0.$$

By means of T we get as the image of C_n the curve

$$y_3^{n-2s}\phi_s(y_1, y_3, y_2^2)+\ldots\ldots=0,$$

* For a comprehensive treatment of birational transformations see K. Doehlemann, "Geometrische Transformationen," Vol. II (1908).

which has s branches touching the line $y_1=0$ at the point $(0, 0, 1)$. If k is odd, $k=2t+1$, the equation of C_n is of the form

$$x_2 x_3^{n-(2t+1)} \phi_t(x_1^2, x_2^2) + \ldots . = 0,$$

and the image of C_n is the curve

$$y_2 y_3^{n-(2t+1)} \phi_t(y_1, y_3, y_2^2) + \ldots . = 0,$$

which has t branches touching each other at the point $(0, 0, 1)$.

The nature of the transformation T requires that the singularities of the curve in the A-plane, barring those in one of the above-discussed exceptional positions, occur in even numbers of similar ones, one pair of which gives rise to one similar singularity on the image of the curve in the A'-plane.

§ 17. *Illustration.*

Let us now determine the genus of a γ_2' on a sextic of genus 5 and not having a g_3', hence having five double points. Since the number of double points is odd, one of them has to be taken in an exceptional position, while the remaining four give rise to two double points on the image curve in the A'-plane. If we take the double point at the fundamental point, the image of the sextic will be a quintic. The sextic cuts the fundamental line in four points besides the fundamental point, hence two branches of the quintic touch each other at the fundamental point in the A'-plane. The latter singularity is equivalent to two double points. The quintic, then, has four double points, and is of genus 2. Algebraically we get the same result by noticing that the equation of a sextic not passing through the center of homology $(1, 0, 0)$ and having a double point at $(0, 1, 0)$ is of the form

$$x_2^4(ax_1^2+bx_3^2) + \ldots . + x_1^6 = 0.$$

Under T it goes over into the quintic

$$y_2^4(ay_1+by_3) + \ldots . + y_1^3 y_3^2 = 0,$$

which has two consecutive double points at $(1, 0, 0)$.

Had we taken the double point on the axis of homology, the result would be the same. In fact consider a sextic with a double point at $(0, 0, 1)$. It cuts the fundamental line in six points. Its image is a sextic having three branches touching each other at the fundamental point in the A'-plane. This singularity is the equivalent of six double points. The sextic in the A'-plane has eight double points, it is of genus 2.

Consider now the case when the normal form of a curve of genus 5 is a quintic with one double point. The center of homology is on the quintic, and if we take the double point as the point $(0, 1, 0)$, the image of the quintic is a quartic with one double point, hence of genus 2.

An involution of order 2 *on a curve of genus* 5 *is of genus* 2.

§ 18. *General Involutions of Order* 2.

We can now proceed to the general case. Given C_n. In order to possess a γ'_2 its equation must remain invariant under some birational transformation of period 2. If the curve is non-singular, hence of maximum genus $=\frac{(n-1)(n-2)}{2}$, the transformation must be linear.* If n is even, the equation of the curve is of the form

$$x_1^n+x_1^{n-2}\phi_2(x_2, x_3)+\ldots.+\phi_n(x_2, x_3)=0.$$

Its image under T is

$$y_1^{\frac{n}{2}}y_3^{\frac{n}{2}}+y_1^{\frac{n-2}{2}}y_3^{\frac{n-2}{2}}\phi_2(y_2, y_3)+\ldots.+\phi_n(y_2, y_3)=0,$$

a curve of order n with two consecutive $\frac{n}{2}$-fold points at the point (1, 0, 0). The genus of the curve is

$$\frac{(n-1)(n-2)}{2}-\frac{n}{2}\left(\frac{n}{2}-1\right)=\frac{(n-2)^2}{4}.$$

Let us apply as a check Zeuthen's formula. The number of coincidences of γ'_2 is n, the points of intersection of $x_1=0$ with the curve,

$$\therefore n=2\left[\frac{(n-1)(n-2)}{2}-1\right]-4(\pi-1)\,;\quad \therefore \pi=\frac{(n-2)^2}{4}.$$

If we denote $\frac{n}{2}$, the number of groups of γ'_2 on a line through the center of homology by r, we can write the genus of the involution in the form $(r-1)^2$.

The genus of γ'_2 on a non-singular curve of even order is $(r-1)^2$, where r is the number of groups of γ'_2 on a line passing through one group of the involution.

If the order of the curve is odd, its equation is of the form

$$x_1^{n-1}\phi_1(x_2, x_3)+x_1^{n-3}\phi_3(x_2, x_3)+\ldots.+\phi_n(x_2, x_3)=0.$$

Its image in the A'-plane is

$$y_1^{\frac{n-1}{2}}y_3^{\frac{n-1}{2}}\phi_1(y_2, y_3)+\ldots.+\phi_n(y_2, y_3)=0,$$

a curve of order n. The genus of the curve is $\frac{(n-1)(n-3)}{4}=r(r-1)$, where $r=\frac{n-1}{2}$, and is, as above, the number of groups of γ'_2 on a line containing one group.

* V. Snyder, "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1909).

The coincidences of γ'_2 are the n points of intersection of $x_1=0$ with the curve and the point $(1, 0, 0)$. By Zeuthen's formula,

$$n+1=2\cdot\left[\frac{(n-1)(n-2)}{2}-1\right]-4(\pi-1), \quad \therefore \pi=\frac{(n-1)(n-3)}{4}.$$

The genus of γ'_2 on a non-singular curve of odd order is $r(r-1)$, where r is the number of groups of γ'_2 on a line passing through one group of the involution.

Thus, on a non-singular quartic γ'_2 is of genus $(2-1)^2=1$, on a non-singular quintic of genus $2\cdot(2-1)=2$, on a non-singular sextic of genus $(3-1)^2=4$, and so on.

If the given curve $F(x)=0$ is non-hyperelliptic, and has the equivalent of not more than $E\left(\frac{n-1}{2}\right)^2-3$ double points, $E(k)$ being the largest integer less than k, it remains invariant under linear transformations only.*

If the number of singularities is greater, the curve may remain invariant under transformations of period 2 other than linear. In that case, however, there exists a curve birationally equivalent to $F(x)=0$, the equation of which contains only even powers of one of the variables, say x_1, $f(x_1^2, x_2, x_3)\equiv f^2(x)=0$, and which is therefore transformed into itself by L.†

Consequently, a curve possessing a γ'_2 is either itself invariant under L or can be put into $(1, 1)$ correspondence with a curve which is invariant under L. In the first instance, as we have seen, if $F(x)=0$ is not hyperelliptic, the involution is multiple. If the involution defines on $F(x)=0$ a transformation which is not linear, consider $f^2(x)=0$, identical with $F(x)=0$ as regards involutions. The pencil of lines through the center of homology cuts the curve in k pairs of points of the involution. If $k=1$, the involution is a rational g'_2, and the curve is, therefore, hyperelliptic. If $k>1$, the involution is multiple. Hence we can generalize the theorem stated in § 15:

A curve having a simple γ'_2 is hyperelliptic. If a non-hyperelliptic curve has a γ'_2 the involution is multiple.

We can readily determine the genus of a γ'_2 on a curve of given characteristics. Consider $F(x)=0$, of genus p. The singularities of the curve, not in exceptional positions, appear in pairs of similar ones. They are equivalent to $2\sum_{\substack{i=2\ldots\\ v=1\ldots}} v_i\cdot\frac{i(i-1)}{2}$ double points, where $2v_i$ denotes the number of i-fold points.

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* V. Snyder, "On Birational ....," American Journal of Mathematics, Vol. XXX (1909).

† A. Hurwitz, "Ueber diejenigen ....," *Math. Annalen*, Vol. XXXII.

They give rise on $f(y)=0$ in the A'-plane to singularities equivalent to $\sum_{\substack{i=2\ldots\\ v=1\ldots}} v_i\cdot\frac{8i(i-1)}{2}$ double points. If $F(x)=0$ has no other singularities, and does not pass through the center of homology $(1, 0, 0)$, $f(y)=0$ will in addition have $\frac{n}{2}$ branches touching each other at $(1, 0, 0)$. Its genus, and therefore the genus of the involution, is

$$\pi=\frac{2(n-1)(n-2)-2\Sigma v_i\cdot i(i-1)-n(n-2)}{4}.$$

The discussion in § 15 has taken cognizance of all possible positions of singularities on $F(x)=0$. When the singularities of $F(x)=0$ are known the determination of the genus of the involution becomes a matter of numerical calculations.

§ 19. *Cyclic Involutions of Any Order.*

The transition to involutions of any order follows directly, if the involution determines a birational transformation of the curve into itself. If a point P goes into P_1, P_1 into P_2, ... and $P_a=P$ by the transformation, the involution is called *cyclic.* The transformation is always periodic, and is either linear, or the given curve $F(x)=0$ can be put into $(1,1)$ correspondence with a curve, the equation of which is of the form $f(x_1^a, x_2, x_3)\equiv f^a(x)=0$, and is therefore invariant under the linear transformation:

$$U\equiv\begin{pmatrix} x_1 & x_2 & x_3 \\ \theta x_1 & x_2 & x_3 \end{pmatrix}, \quad \theta^a=1.*$$

If $F(x)=0$ has a γ'_a so does $f^a(x)=0$. The groups of the involution on $f^a(x)=0$ are cut out by a pencil of lines having its vertex at the center of homology $(1, 0, 0)$. If the lines of the pencil cut out only one group, γ'_a is a rational g'_a; if they cut the curve in more than one group the involution is multiple. Hence,

A cyclic involution of order a on a given curve is either rational or, if irrational, is multiple.

A multiple cyclic involution may also be rational.

The genus of γ'_a can be determined in a manner exactly analogous to the one employed in the determination of the genus of an involution of order 2. The singularities of $F(x)=0$, not in exceptional positions, will have to appear in groups of a similar ones giving rise to $1/a$-th, their number of similar singularities on $f(y)=0$. If $F(x)=0$ does not pass through the point $(1,0,0)$, $f(y)=0$ has $\frac{n(a-1)}{2}$ branches touching each other at $(1, 0, 0)$, and so on.

* A. Hurwitz, "Ueber diejenigen ....," *Math. Annalen*, Vol. XXXII.

§ 20. *Non-Cyclic Involutions.*

Given a cone $K_{\psi,\pi}=0$ of order Ψ, genus π and a surface $F_a=0$. The curve C of intersection goes b times through the vertex of the cone and is of order m, genus p, where $m=a\Psi+b$ and p is determined from Sturm's formula *

$$p=a(a-1)\Psi+(a-1)(b-1)+a\pi.$$

The curve C possesses an involution of order a, genus π. Through every point P of C passes a generator of K, which meets C in $a-1$ points besides P. But each generator of K meets but one group of a points, hence K, C are in $(1, a)$ correspondence.

If C is projected on a plane section γ of K, when the center of projection is at the vertex of the cone, C is projected a-fold on γ. If the center of projection 0 is on C but not at the vertex of the cone, C will be projected into a plane curve C_1 of order $m-1$, having $a-1$ branches with a common tangent at 0_1 the point in which the generator of K through 0 pierces the plane of projection. Let P be any point on C. The generator through P and the point 0 determine a plane which passes through the vertex of K, hence cuts it in $\Psi-1$ generators besides the one through 0. The plane meets the plane of projection in a line through 0_1 and contains $\Psi-1$ groups of the involution. We have therefore an illustration of a multiple non-cyclic involution on C_1. The image $\gamma'_{a,\pi}$ may be assumed at will, hence curves having involutions of any order can be constructed, which have a given curve for image of the involution.

The number of coincidences is the number of tangents to C through the vertex of the cone K. By the Cayley-Brill formula this is seen to be $2(a-1+p)$. This, by Castelnuovo's formula already cited (§ 6), is the maximum number an involution $\gamma'_{a,\pi}$ can have.

If the center of projection is on K but not on C the conditions are unchanged except that the vertex of the pencil of lines in the plane of C_1 is now an a-fold point at which all a branches have a common tangent. As before, each line of the pencil contains $\Psi-1$ groups of the involution.

Finally, if the center of projection 0 is not on K, the vertex 0_1 of the plane pencil is not on C_1, and each line of the pencil contains Ψ groups of the involution.

Next, suppose we have a ruled surface $R_{n,\pi}=0$ of order n, and π the genus of a plane section Γ. If $C_m=0$ is a complete intersection of R and $F_k=0$, $m=kn$, its characteristics are connected with those of R by the following relation:

$$m(k+n-2)=r+2d, \tag{10}$$

* R. Sturm, "Ueber das Geschlecht von Curven auf Kegeln," *Math. Annalen*, Vol. XIX (1882), pp. 487-488.

where d is the order of the double curve on R and is given by

$$d=\frac{(n-1)(n-2)}{2}-\pi,$$

and r, the rank of C is $r=2m+2p-2$. Substituting the values of r and d in (10) we get

$$m(k-1)-p-\delta=\frac{k(k-1)n}{2}-k(\pi-1)-1, \tag{11}$$

δ being the number of times R and F touch.

If the complete intersection of R and F is made up of C and s generators, so that $m=kn-s$, we have

$$(m-s)(k+n-2)=r-r'.\text{*}$$

r', the rank of the system of generators, is zero, hence

$$(kn-2s)(k+n-2)=2(kn-s)+2p-2+k(n-1)(n-2)-2k\pi-2s(n-2),$$

or, as above,

$$m(k-1)-p-\delta=\frac{k(k-1)n}{2}-k(\pi-1)-1. \tag{11}$$

If the residual of C_m is another curve $C_{m'}$, and if there exists an $F_{k'}=0$ which cuts R in C_m or $C_{m'}$ and s generators then, by means of relation (11) we can find the genus of C_m or $C_{m'}$ in terms of the characteristics of R, then the genus of the other curve by means of (10), which reduces to the form of (11). Formula (11), which is a generalization of Sturm's formula due to Segre,† is applicable to any curve on a ruled surface.

If now C is projected into the plane curve C_1, the k points in which each generator of R meets C will be projected into a group of k collinear points, but the same line contains $m-k$ other points, not belonging to a group. The lines containing each a group of the involution envelope a curve of class n, birationally equivalent to the dual of Γ. Each of the remaining points on a line, not belonging to the group on that line, belongs to a group on another tangent to the envelope.

In case the surface $F=0$ is also a ruled surface, on C are two distinct involutions, and hence also on C_1. Let $F\equiv R'_{n',\pi'}=0$ and $R_{n,\pi}=0$ have j generators in common, so that $m=nn'-j$, $\delta=2j$, $k=n'$, $k'=n$. From Segre's formula we have

$$(k-1)(nn'-j)-p-2j=\frac{k(k-1)n}{2}-k(\pi-1)-1,$$

$$(n'-1)(nn'-j)-p-2j=\frac{k'(k'-1)n'}{2}-k'(\pi'-1)-1.$$

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* Salmon, "Analytic Geometry of Three Dimensions," fifth edition, Vol. I (1912), § 346, p. 358.

† C. Segre, "Recherches générales sur les courbes et les surfaces réglées algébriques," *Math. Annalen*, Vol. XXXIV (1889), pp. 1-25.

The number of coincidences in the first involution is $2(n'-1+p)$ and in the second is $2(n-1+p)$. The maximum genus of C when $j=0$ can be obtained from Salmon's theory of postulation.

§ 21. *Restrictions on the Moduli of a Curve Having an Involution.*

The application of the methods used in the preceding pages to the second question we set out to answer is immediate. Let us first consider a simple case. Given an involution of order 2 and genus 1, to find the simplest curve that can possess it; in other words, find the simplest curve upon which a non-singular cubic can be mapped doubly. Consider the cubic

$$y_2^2 y_3 = \phi_3(y_1, y_3). \tag{12}$$

By means of T^{-1} we find as its image in the A-plane the sextic

$$x_1^2 x_3^4 = \phi_3(x_1^2, x_3^2). \tag{12'}$$

The sextic has a four-fold point and its genus is therefore 2. We notice that (12′) is invariant not only under L, but also under another homology which replaces x_3 by $-y_3$, or that the existence of one elliptic involution of order 2 on a curve of genus 2 necessitates the existence of another involution of the same order and genus. In the same manner most of the theorems established by Torelli* by transcendental methods, can be proved as simple corollaries of the preceding theorems.

In general, in order to find the genus of a curve which possesses a $\gamma'_{a,\pi}$, we start in the A'-plane with a normal form of a curve of genus π. Every multiple point of the curve, not in an exceptional position, gives rise to a similar multiple point on the image curve in the A-plane. The procedure laid down in § 16 is followed in the determination of correspondents of multiple points in exceptional positions. The genus of the image curve is then easily calculated.

* R. Torelli, "Sulle curve di genere due contenenti una involuzione ellitica," *Rend. Acc. Napoli*, s. 3, Vol. XVII (1911), pp. 412–419.

The Set of Eight Self-Associated Points in Space.

By John Rogers Musselman.

Introduction.

Associated point sets were first discussed by Rosanes * and Sturm.† Later a type of self-conjugate association was treated by Study. ‡ Recently Coble § discussed the association of a set P_n^k (n-points in S_k) with a set Q_n^{n-k-2} (n-points in S_{n-k-2}) and derived the complete systems of invariants for P_6^1 and P_6^2.

An interesting case of associated sets occurs when both sets of points are in the same space. The term association as defined implies a mutual ordering of the points. It may be possible to project the one set upon the other in the order of association. The two sets are then said to be self-associated, and the order will be referred to as the identical order. For the P_8^3 this is the well-known set of base points of a net of quadrics. If the associated sets can be projected one upon the other in some order other than the identical, the sets are said to be self-associated in other than the identical order. It is this type of P_8^3 which will be discussed in this paper. In § 2 the general set of eight points self-associated in some order is treated. If this self-association requires that the set be the base points of a net of quadrics, the set is said to be of type B and is discussed in § 3. If the eight points lie on a rational space cubic, the set is of type R and is considered in § 4. Various theorems and facts of immediate use are grouped in § 1. In order to restrict the number of cases, no order of self-association is discussed if in the set two points should coincide, three lie on a line, or four be in a plane. || Such a set is said to be of type E.

Those sets of points which can be self-associated in other than the identical order, can naturally be determined only to within projective transformation.

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* *Crelle*, Bd. LXXXVIII (1880), p. 241.

† *Math. Ann.*, Bd. I (1869), p. 533, and Bd. XXII (1883), p. 569.

‡ *Math. Ann.*, Bd. LX (1905), p. 321.

§ *Transactions*, Vol. XVI (1915), p. 155. This paper hereafter will be cited as C.

|| This excludes some cases of interest. See footnote on Desmic Systems, p. 81.

Most of the conclusions in § 2 are given in terms of the elliptic parameters of the points; if the curve on them degenerates, the geometrical construction of the set is given. In § 3 the ten types of the self-projective planar quartic given by Wiman* are tabulated and their connection, where possible, shown with specific orders of self-association of the P_8^3. The types of self-associated sets on a rational cubic and the groups connected with them are given in § 4.

§ 1. Let the set of eight points in space be given by the equations

$$(up_1)=0, \quad (up_2)=0, \ldots, (up_8)=0.$$

Since any five points in space are linearly related, these equations are connected by four linear relations. Let them be

$$q_{1i}(up_1)+q_{2i}(up_2)+\ldots+q_{8i}(up_8)\equiv 0, \qquad i=1, 2, 3, 4.$$

Multiplying them respectively by v_i and adding, we have the single identity in u and v,

$$(1) \qquad (vq_1)(up_1)+(vq_2)(up_2)+\ldots+(vq_8)(up_8)\equiv 0,$$

which leads to the set Q_8^3. Thus a set P_8^3 projectively defines an associated set Q_8^3, and the relation is mutual. If the set P_8^3 is self-associated the above identity can be written as

$$(2) \qquad \lambda_1(up_1)(vq_1)+\lambda_2(up_2)(vq_2)+\ldots+\lambda_8(up_8)(vq_8)\equiv 0,$$

where the points q_i are merely some permutation of the points p_i. The latter can be replaced by the two following identities:

$$(3) \qquad \begin{aligned} &\lambda_1(up_1)(uq_1)+\lambda_2(up_2)(uq_2)+\ldots+\lambda_8(up_8)(uq_8)\equiv 0,\\ &\lambda_1(p_1q_1x) \quad +\lambda_2(p_2q_2x) \quad +\ldots+\lambda_8(p_8q_8x) \quad \equiv 0. \end{aligned}$$

We shall have need of the following facts in the later paragraphs:

(4) If the associated sets P_8^3 and Q_8^3 in S_3 be placed so that the first five points of each constitute the same base, the remaining three points of each set lie in the same plane α, and form polar triangles of the conic Q_α, in α, which is apolar to all sections by α of the basic quadrics.†

(5) If by projection from one of the points q, say q_8, and section by a plane, there is obtained from the remaining seven points $q_1, \ldots, q_7$ a set Q_7^2, this set is associated with the set P_7^3 formed by the points $p_1, \ldots, p_7$ of P_8^3.‡ From this theorem is obtained a result of importance for any order of association containing a cycle of two letters, say (p_7p_8). By projection from p_8 we get seven points p in a plane associated with the set of seven points q. If the

* *Math. Ann.*, Bd. XLVIII (1897), p. 222. † C., p. 159. ‡ C., p, 158.

sets P_8^3 and Q_8^3 are not self-associated, p_7 will have been sent into q_8. Projecting now from q_8 we obtain six points q in a plane associated with six points p in the plane. If the sets P_8^3 and Q_8^3 are self-associated, then p_7 is q_8, and the projected sets in the plane coincide and are self-associated. The set of six self-associated points in the plane has been fully treated by Coble,* and his conclusions are available for our purposes.

(6) Two sets associated with a third set are projective; for the sets are only projectively known so that a third set will projectively define its associated set.

There are twenty-one possible orders in which a P_8^3 may be self-associated, as (12), (12)(34), ..., where the points are indicated by their subscripts and where, for example, by (12345) we mean $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ is associated with $p_2, p_3, p_4, p_5, p_1, p_6, p_7, p_8$.

§ 2. In this section we treat those orders of self-association of the P_8^3 which permit the eight points to lie on a unique elliptic quartic. Three theorems are now proved which enable us to classify some of the orders of self-association as belonging to types B and E; such types are discussed later.

(7) *Any order of association containing only one cycle of three points is of type E.* Suppose the cycle to be (123)...., where the other five points enter in any possible arrangement save a cycle of three. By applying (6) this order of association (123).... implies projectivity in the order (312)...., and, consequently, is projective in that power of this order which is the least common multiple of the periods of the cycles—not including the cycle of three points. The original order of association is then projective to the order (123), let us say. The transformation defined by this projectivity has multipliers 1, ω, or ω^2; ($\omega^3=1$). Evidently two multipliers at least will be alike and we can choose these two to be unity. There are then three types to consider: 1, 1, 1, ω; 1, 1, ω, ω; 1, 1, ω, ω^2. The transformation having the first set of multipliers has a fixed point and a fixed plane; the one with the second set of multipliers has two lines of fixed points, while the transformation with the third set of multipliers has a line of fixed points and a line of fixed planes. In all these types we can not have five fixed points without four of the P_8^3 lying in a plane and thus giving a P_8^3 of type E. *Hence the following orders* (123), (123)(45), (123)(45)(67), (123)(4567) *and* (123)(45678) *are of type E and will not be discussed in this paper.*

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* C., p. 163.

(8) *Any order of association containing only one cycle of four points is of type E.* By an argument similar to that in (7) it can be shown that the order of association (1234).... implies projectivity to the order of association (13)(24). This type of projective transformation is the well-known harmonic perspectivity in a point and plane, or in two lines. In either case to have four fixed points would require four of the P_8^3 to lie in a plane; and so any order of association containing only one cycle of four points leads to a P_8^3 of type *E*. *Hence the orders of association* (1234), (1234)(56) *and* (1234)(56)(78) *are of type E and will not be considered in this paper.*

(9) *All further orders of association of odd period are of type B.* In fact all orders of association of odd period are of type *B*, but as some are included in the statements of the last two paragraphs, we shall say only those orders of odd period, not previously excluded, are of type *B*. Let Π be the order of association. By (6) the order of association Π implies projectivity to the order of association Π^2. To be of type B, Π is projective to itself in the identical order, or $\Pi^{2n}=\Pi$. This says $\Pi^{2n-1}=1$ which is true if Π is of odd period. Hence all orders of association of odd period are of type *B*, *and the following orders* (123)(456), (12345) *and* (1234567) *will be reserved for discussion in the next section of this paper.*

(10) The remaining ten orders of self-association will now be taken up in detail, considering first the order (123456)(78). This order implies projectivity to the order (135)(246); thus defining a transformation with multipliers taken from $1, \omega, \omega^2$ ($\omega^3=1$). Let us give the P_8^3 the following coordinates:

$1:\ 1,\ 1,\ 1,\ 1,\quad 2:\ \lambda,\ 1,\ x_3,\ x_4,\quad 3:\ 1,\ 1,\ \omega^2,\ \omega,\quad 4:\ \lambda,\ 1,\ \omega^2 x_3,\ \omega x_4,$

$5:\ 1,\ 1,\ \omega,\ \omega^2,\quad 6:\ \lambda,\ 1,\ \omega x_3,\ \omega^2 x_4,\quad 7:\ 1,\ 0,\ 0,\ 0,\quad 8:\ 0,\ 1,\ 0,\ 0.$

The sixteen equations resulting from the identity

$$\lambda_1(up_1)(vp_2)+\ldots\ldots+\lambda_6(up_6)(vp_1)+\lambda_7(up_7)(vp_8)+\lambda_8(up_8)(vp_7)\equiv 0$$

have the following matrix where the row $[ij]$ indicates the coefficients of the equation derived from the coefficient of $u_i v_j$ $(i, j=1, \ldots, 4)$.

$$\begin{array}{llllllll l}
[11] & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 0 & 0, \\
[22] & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0, \\
[33] & x_3 & \omega^2 x_3 & \omega x_3 & x_3 & \omega^2 x_3 & \omega x_3 & 0 & 0, \\
[44] & x_4 & \omega x_4 & \omega^2 x_4 & x_4 & \omega x_4 & \omega^2 x_4 & 0 & 0, \\
[12] & 1 & \lambda & 1 & \lambda & 1 & \lambda & 1 & 0, \\
[21] & \lambda & 1 & \lambda & 1 & \lambda & 1 & 0 & 1, \\
[13] & x_3 & \lambda\omega^2 & \omega^2 x_3 & \lambda\omega & \omega x_3 & \lambda & 0 & 0, \\
[31] & \lambda & x_3 & \lambda\omega^2 & \omega^2 x_3 & \lambda\omega & \omega x_3 & 0 & 0,
\end{array}$$

[14]	x_4	$\lambda\omega$	ωx_4	$\lambda\omega^2$	$\omega^2 x_4$	λ	0	0,
[41]	λ	x_4	$\lambda\omega$	ωx_4	$\lambda\omega^2$	$\omega^2 x_4$	0	0,
[23]	x_3	ω^2	$\omega^2 x_3$	ω	ωx_3	1	0	0,
[32]	1	x_3	ω^2	$\omega^2 x_3$	ω	ωx_3	0	0,
[24]	x_4	ω	ωx_4	ω^2	$\omega^2 x_4$	1	0	0,
[42]	1	x_4	ω	ωx_4	ω^2	$\omega^2 x_4$	0	0,
[34]	x_4	ωx_3	x_4	ωx_3	x_4	ωx_3	0	0,
[43]	x_3	$\omega^2 x_4$	x_3	$\omega^2 x_4$	x_3	$\omega^2 x_4$	0	0.

We can use [12] and [21] to determine λ_7 and λ_8 after the other equations have been satisfied. Both [33] and [44] can be divided respectively by x_3 and x_4 (which can not be zero else four points in a plane); using them in connection with [22] to eliminate $\lambda_1, \lambda_3, \lambda_5$ from the other equations we get the following five equations in $\lambda_2, \lambda_4, \lambda_6$:

$$(x_3-\omega\lambda)(\lambda_2+\omega^2\lambda_4+\omega\lambda_6)=0, \qquad (x_4-\omega^2)(\lambda_2+\omega\lambda_4+\omega^2\lambda_6)=0,$$
$$(x_3-\omega)(\lambda_2+\omega^2\lambda_4+\omega\lambda_6)=0, \qquad (\omega x_3-x_4)(\lambda_2+\lambda_4+\lambda_6)=0.$$
$$(x_4-\omega^2\lambda)(\lambda_2+\omega\lambda_4+\omega^2\lambda_6)=0,$$

If $\lambda_2+\lambda_4+\lambda_6=0$, and $\lambda_2+\omega^2\lambda_4+\omega\lambda_6=0$, then $\lambda_2+\omega\lambda_4+\omega^2\lambda_6\neq 0$, $x_4=\omega^2$. $\lambda=1$.

If $\lambda_2+\lambda_4+\lambda_6=0$, and if $\lambda_2+\omega\lambda_4+\omega^2\lambda_6=0$, then $\lambda_2+\omega^2\lambda_4+\omega\lambda_6\neq 0$, and $x_3=\omega$ and $\lambda=1$.

But $\lambda\neq 1$, else six points of the P_8^3 are in a plane whence

$$\lambda_2+\omega\lambda_4+\omega^2\lambda_6=0, \qquad \lambda_2+\omega^2\lambda_4+\omega\lambda_6=0,$$
$$\lambda_2+\lambda_4+\lambda_6\neq 0, \qquad \text{and} \qquad x_4=\omega x_3.$$

With this relation on the constants, the set P_8^3 has the following coordinates:

1: 1, 1, 1, 1, 2: a, 1, b, ωb, 3: 1, 1, ω^2, ω, 4: a, 1, $\omega^2 b$, $\omega^2 b$,
5: 1, 1, ω, ω^2, 6: a, 1, ωb, b, 7: 1, 0, 0, 0, 8: 0, 1, 0, 0.

But points 7, 8, 1, 4; 7, 8, 2, 5; 7, 8, 3, 6 lie in planes so the order of association (123456) (78) leads to a P_8^3 of type E and will not be discussed further.

(11) The application of theorem (4) shows that the order of self-association (12) is possible. Choosing 4, 5, 6, 7, 8 as the base points we have to require that $\overline{13}$ and $\overline{23}$ be tangent lines of the conic Q_a. Analytically we see that, from the two identities,

$$(\lambda_1+\lambda_2)(up_1)(up_2)+\lambda_3(up_3)^2+\ldots\ldots+\lambda_8(up_8)^2\equiv 0,$$
$$(\lambda_1-\lambda_2)(p_1p_2x)\equiv 0,$$

unless two points coincide, $\lambda_1=\lambda_2$. Quadrics on the six points 3, 4, ..., 8 must cut the line $\overline{12}$ harmonically. There are four independent quadrics on

six points, so 1 and 2 must be apolar to these quadrics and are therefore corresponding points of the Weddle surface determined by the other six points. The quartic curve on the P_8^3 has degenerated, for 1 and 2 lie on a bisecant of the cubic curve through the other six points. This set contains five absolute constants.

(12) The second identity for the order of association (12)(34) requires $\lambda_1=\lambda_2$ and $\lambda_3=\lambda_4$ unless two points coincide, or four lie on a line. Using points 5, 6, 7, 8 as reference points, 4 as unit point and 1, 2, 3 as x, y, z, respectively, the ten equations resulting from the first identity are:

$$\begin{aligned}
&2\lambda_1x_1y_1+2\lambda_3z_1+\lambda_5=0, && \lambda_1(x_1y_2+x_2y_1)+\lambda_3(z_1+z_2)=0,\\
&2\lambda_1x_2y_2+2\lambda_3z_2+\lambda_6=0, && \lambda_1(x_1y_3+x_3y_1)+\lambda_3(z_1+z_3)=0,\\
&2\lambda_1x_3y_3+2\lambda_3z_3+\lambda_7=0, && \lambda_1(x_1y_4+x_4y_1)+\lambda_3(z_1+z_4)=0,\\
&2\lambda_1x_4y_4+2\lambda_3z_4+\lambda_8=0,
\end{aligned}$$

$$\begin{aligned}
\lambda_1(x_2y_3+x_3y_2)+\lambda_3(z_2+z_3)=0,\\
\lambda_1(x_2y_4+x_4y_2)+\lambda_3(z_2+z_4)=0,\\
\lambda_1(x_3y_4+x_4y_3)+\lambda_3(z_3+z_4)=0.
\end{aligned}$$

Eliminating λ_1, λ_3z_1, λ_3z_2, $-\lambda_3z_4$ we obtain:

$$\begin{aligned}
x_1(y_3-y_4)+x_2(y_4-y_3)+x_3(y_1-y_2)+x_4(y_2-y_1)=0,\\
x_1(y_2-y_3)+x_2(y_1-y_4)+x_3(y_4-y_1)+x_4(y_3-y_2)=0,\\
x_1(y_4-y_2)+x_2(y_3-y_1)+x_3(y_2-y_4)+x_4(y_1-y_3)=0.
\end{aligned}$$

If these three equations can be satisfied, the ten equations above can likewise be satisfied. These three equations represent for given y a pencil of planes, so x runs along a line. From their symmetry for given x, y is on a line. Hence this P_8^3 will contain four absolute constants.

The relations connecting x and z are:

$$\begin{aligned}
x_2x_4(z_1+z_3)-x_2x_3(z_1+z_4)-x_1x_4(z_2+z_3)+x_1x_3(z_2+z_4)=0, \quad A\\
x_2x_4(z_1+z_3)-x_3x_4(z_1+z_2)-x_1x_2(z_3+z_4)+x_1x_3(z_2+z_4)=0, \quad B\\
x_1x_2(z_3+z_4)-x_1x_4(z_2+z_3)-x_2x_3(z_1+z_4)+x_3x_4(z_1+z_2)=0, \quad C
\end{aligned}$$

three of the pencil of quadrics containing the quartic on the eight points. These surfaces can easily be constructed for A has for generators the lines $\overline{56}$, $\overline{78}$, and the line joining 3 to the fourth harmonic of 4 as to $\overline{56}$, $\overline{78}$. Similarly, B has for generators $\overline{58}$, $\overline{67}$, and the line joining 3 to the fourth harmonic of 4 as to $\overline{58}$, $\overline{67}$; while the generators of C are the lines $\overline{57}$, $\overline{68}$, and the join of 3 to the fourth harmonic of 4 as to $\overline{57}$, $\overline{68}$. The quadrics are cut in line pairs by the planes of the reference tetrahedron, since its edges are their generators. Thus the plane $x_1=0$ cuts the quadrics respectively in $x_2[x_4(z_1+z_3)-x_3(z_1+z_4)]$, $x_4[x_2(z_1+z_3)-x_3(z_1+z_2)]$ and $x_3[x_4(z_1+z_2)-x_2(z_1+z_4)]$. Three of these lines

are the sides of the reference triangle in the plane $x_1=0$, and the other three are lines through the vertices of the triangle meeting in a point. Moreover, this point is on the quartic curve. For any plane cuts the pencil of quadrics in a pencil of conics on four points of the curve. The conics here are line pairs on the vertices of the reference triangle. They must have another point in common which can not lie on the sides of the triangle, hence the lines through the vertices meet in a point on the curve. We thus obtain four more points on the curve, namely:

$$\begin{array}{lllll} 0, & z_1+z_2, & z_1+z_3, & z_1+z_4, & \alpha \\ z_2+z_1, & 0, & z_2+z_3, & z_2+z_4, & \beta \end{array} \qquad \begin{array}{lllll} z_3+z_1, & z_3+z_2, & 0, & z_3+z_4, & \gamma \\ z_4+z_1, & z_4+z_2, & z_4+z_3, & 0. & \delta \end{array}$$

But 3, 4, α, 5 are in a plane; so also 3, 4, β, 6; 3, 4, γ, 7 and 3, 4, δ, 8.

Calling the elliptic parameters of the points 1, 2, ..., 8, $u_1, u_2, \ldots, u_8$, since 6, 7, 8, α are in a plane,

$$u_3+u_4+u_5\equiv u_6+u_7+u_8, \text{ or } u_3+u_4\equiv -u_5+u_6+u_7+u_8.$$

Similarly, $u_3+u_4\equiv u_5-u_6+u_7+u_8\equiv u_5+u_6-u_7+u_8\equiv u_5+u_6+u_7-u_8$,

whence $\quad 2(u_3+u_4)=4u_5, \text{ or } -u_3-u_4+2u_5=0.$

Also $\quad -u_3-u_4+2u_6=0, \quad -u_3-u_4+2u_7=0, \quad -u_3-u_4+2u_8=0.$

These relations say that the four planes on $\overline{-u_3, -u_4}$, tangent to the curve, are tangents at the points u_5, u_6, u_7, u_8. Find that quadric of the pencil having $\overline{u_3, u_4}$ as generator; a plane through this line will cut the quadric in $\overline{-u_3, -u_4}$. The four planes on this line $\overline{-u_3, -u_4}$ tangent to the curve cut out the points u_5, u_6, u_7, u_8. Points 1 and 2 are treated like 3 and 4. Hence $\overline{12}$, $\overline{34}$ are any two generators of the same system of a quadric on the curve; 5, 6, 7, 8 are the points where the curve is tangent to the generators of this same system. The P_8^3 thus contains four absolute constants, and the conditions on the parameters of the points are

$$u_1+u_2=u_3+u_4=2u_5=2u_6=2u_7=2u_8.$$

(13) The second identity for the order of association (12) (34) (56) (78) is

$$(\lambda_1-\lambda_2)(12x)+(\lambda_3-\lambda_4)(34x)+\ldots+(\lambda_7-\lambda_8)(78x)\equiv 0.$$

If one difference vanishes, say the first, the lines $\overline{34}$, $\overline{56}$, and $\overline{78}$ are on a point, whence four points lie in a plane. If two differences vanish, either two pairs of points coincide or four points lie on a line. If three differences vanish, two points must coincide. These conditions all lead to sets of type E. If none of the differences vanish, the four lines are lines of the same system of generators on a quadric. Moreover, this condition is sufficient because the first

identity can be satisfied by choosing $\lambda_1+\lambda_2=0, \ldots., \ldots., \lambda_7+\lambda_8=0$. This set involves six absolute constants: one for the curve, one for the quadric, and one for each generator. In terms of the elliptic parameters of the points we have

$$u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=k.$$

However the case remains for which all the differences may vanish. Applying theorem (5) and projecting from point 7 (or 8) we get six self-associated points in a plane which under the order of association (12) (34) (56) requires the points to lie on three lines meeting in a point.* This means that looking from point 7 (or 8) the projections of the lines $\overline{12}, \overline{34}, \overline{56}$ upon a plane meet in a point. To do so the line of perspection from 7 (or 8) must meet these lines. The lines of perspection from 7 and 8 are distinct, else four points in a plane. So 7 and 8 lie on two cross generators on the quadric having $\overline{12}, \overline{34}, \overline{56}$ as generators. Either the four lines are generators of the same system on a quadric, or the quadrics having $\overline{12}, \overline{34}, \overline{56}$ and $\overline{34}, \overline{56}, \overline{78}$, respectively, as generators are distinct. Since both are quadrics on the curve, their intersection, $\overline{34}, \overline{56}$ and two cross generators, shows 1, 2, 7, 8 must lie on the cross generators whence four points would be in a plane. Hence the two quadrics must be the same one and $\overline{12}, \overline{34}, \overline{56}, \overline{78}$ are generators of the same system on a quadric.

The first identity, $\lambda_1(u1)(u2)+\ldots.+\ldots.+\lambda_7(u7)(u8)\equiv 0$, says that any quadric apolar to the first three pairs must cut $\overline{78}$ harmonically as to 7 and 8. Examine the quadric made of planes $\overline{134}, \overline{156}$. The plane $\overline{134}$, being tangent along the generator $\overline{34}$, cuts out on the quadric that generator of the other system through the point 1. But the plane $\overline{156}$ cuts out the same generator. Now the points where these lines cut $\overline{78}$, since the generators coincide, must be at 7 or at 8. In either case, four points lie in a plane, and we are led to a P_8^3 of type E. Therefore the order (12) (34) (56) (78) requires the four lines $\overline{12}, \overline{34}, \overline{56}, \overline{78}$ to be lines of the same system on a quadric. The set contains six constants and is given parametrically by

$$u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=k.$$

(14) We can dispose of the order of association (123) (456) (78) in a few words. It implies projectivity to the order (132) (465). It is also associated in the cube of its order, that is, in the order (78). These facts enable us to construct the set. The six points 1, 2, 3, 4, 5, 6 determine a cubic curve on which they lie in two cyclic sets 1, 2, 3 and 4, 5, 6. The transformation sending

* C., p. 161.

these points into each other cyclically has two fixed points on the curve. Join these points, and 7 and 8 are points on the Weddle, determined by the first six points, cut out by this line of fixed points. The quartic curve has degenerated into a cubic and its bisecant, while the P_8^3 contains but one absolute constant.

(15) The order of association (1234) (5678) implies projectivity to the order (13) (24) (57) (68). The points must lie harmonically separated in pairs as to two fixed lines. Let them have the coordinates:

$$\begin{array}{llll} 1:\ 1,\ 0,\ 0,\ a, & 2:\ 0,\ 1,\ b,\ 0, & 3:\ 1,\ 0,\ 0,\ -a, & 4:\ 0,\ 1,\ -b,\ 0, \\ 5:\ 1,\ 1,\ 1,\ 1, & 6:\ x_1,\ x_2,\ x_3,\ x_4, & 7:\ 1,\ 1,\ -1,\ -1, & 8:\ x_1,\ x_2,\ -x_3,\ -x_4. \end{array}$$

Substituting these values in the identity we obtain sixteen equations which can be satisfied if the following conditions are satisfied:

$$x_4^2+x_1^2=x_3^2+x_2^2=x_1x_2+x_3x_4=x_1x_3-x_2x_4=0, \quad (ax_3-bx_2)^2+(ax_1+bx_4)^2=0.$$

The first four relations are satisfied by $x_1=ix_4$, $x_3=-ix_2\,[i^4=1]$. Putting these values in the remaining condition,

$$(-iax_2-bx_2)^2+(aix_4+bx_4)^2=0, \text{ or } (x_2^2+x_4^2)(ia+b)^2=0,$$

whence either $$x_2^2+x_4^2=0, \text{ or } a=ib.$$

CASE I. $x_1=ix_4$, $x_3=-ix_2$ and $x_2=\pm ix_4$.

If $x_2=ix_4$, letting $x_4=1$, we have $x_1=i$, $x_2=i$, $x_3=1$, $x_4=1$. With these values points 5, 6, 7, 8 are in a plane, so they lead to a P_8^3 of type E.

If $x_2=-ix_4$, letting $x_4=1$, we have $x_1=i$, $x_2=-i$, $x_3=-1$, $x_4=1$, which gives a P_8^3 involving two constants, and whose coordinates are:

$$\begin{array}{llll} 1:\ 1,\ 0,\ 0,\ a, & 2:\ 0,\ 1,\ b,\ 0, & 3:\ 1,\ 0,\ 0,\ -a, & 4:\ 0,\ 1,\ b,\ 0, \\ 5:\ 1,\ 1,\ 1,\ 1, & 6:\ 1,\ -1,\ i,\ -i, & 7:\ 1,\ 1,\ -1,\ -1, & 8:\ 1,\ -1,\ -i,\ i. \end{array}$$

CASE II. $x_1=ix_4$, $x_3=-ix_2$, $a=ib$ gives a P_8^3 containing two absolute constants, whose coordinates are:

$$\begin{array}{llll} 1:\ 1,\ 0,\ 0,\ ib, & 2:\ 0,\ 1,\ b,\ 0, & 3:\ 1,\ 0,\ 0,\ -ib, & 4:\ 0,\ 1,\ -b,\ 0, \\ 5:\ 1,\ 1,\ 1,\ 1, & 6:\ 1,\ -ic,\ -c,\ -i, & 7:\ 1,\ 1,\ -1,\ -1, & 8:\ 1,\ -ic,\ c,\ i. \end{array}$$

Both P_8^3's give the lines $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ as generators of the same system of a quadric on the curve.

In Case I the four generators are harmonic, and the double-ratio of the generators through 1, 2, 5, 6 on the edge $\overline{e_1e_4}$ of the reference tetrahedron is

$$\frac{(a-1)(b+i)}{(b-1)(a+i)}.$$

In Case II the double-ratio of $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ is ic, and the double-ratio of the generators, on $\overline{e_1e_4}$, through the points 1, 2, 5, 6 is $\dfrac{(b+i)^2}{b^2-1}$.

(16) The order of association (12345)(67) implies projectivity to the order (13524). Let P_8^3 be

$$1: 1,\ 1,\ 1,\ 1, \qquad 2: 1,\ \epsilon^{3a},\ \epsilon^{3b},\ \epsilon^{3c}, \qquad 3: 1,\ \epsilon^{a},\ \epsilon^{b},\ \epsilon^{c}, \qquad 4: 1,\ \epsilon^{4a},\ \epsilon^{4b},\ \epsilon^{4c},$$
$$5: 1,\ \epsilon^{2a},\ \epsilon^{2b},\ \epsilon^{2c}, \qquad 6: 1,\ 0,\ 0,\ 0, \qquad 7: 0,\ 1,\ 0,\ 0, \qquad 8: 0,\ 0,\ 1,\ 0,$$

where a, b, c are all different and different from zero, and ϵ^a is a fifth root of unity.

The identity (2) for the order of association (12345)(67) is:

$$\lambda_1(u1)(v2)+\ldots.+\lambda_5(u5)(v1)+\lambda_6(u6)(v7)+\ldots.+\lambda_8(u8)(v8)\equiv 0.$$

The coefficients of u_1v_2, u_2v_1, u_3v_3 determine λ_6, λ_7, λ_8, respectively. The remaining coefficients contain only the first five λ's and reduce to eight equations whose matrix is:

$$\begin{array}{ccccc}
1 & \epsilon^{a} & \epsilon^{2a} & \epsilon^{3a} & \epsilon^{4a}, \\
1 & \epsilon^{c} & \epsilon^{2c} & \epsilon^{3c} & \epsilon^{4c}, \\
1 & 1 & 1 & 1 & 1, \\
1 & \epsilon^{3b} & \epsilon^{b} & \epsilon^{4b} & \epsilon^{2b}, \\
1 & \epsilon^{3c} & \epsilon^{c} & \epsilon^{4c} & \epsilon^{2c}, \\
1 & \epsilon^{3(a+b)} & \epsilon^{(a+b)} & \epsilon^{4(a+b)} & \epsilon^{2(a+b)}, \\
1 & \epsilon^{3(a+c)} & \epsilon^{(a+c)} & \epsilon^{4(a+c)} & \epsilon^{2(a+c)}, \\
1 & \epsilon^{3(b+c)} & \epsilon^{(b+c)} & \epsilon^{4(b+c)} & \epsilon^{2(b+c)}.
\end{array}$$

Let ϵ^d be the remaining fifth root of unity not used in the coordinates of the points. We have two possibilities.

CASE I. $a+b=5$, $c+d=5$, CASE II. $a+c=5$, $b+d=5$,

Iα. $a=1$, $b=4$, $c=2$, $d=3$, IIα. $a=1$, $b=2$, $c=4$, $d=3$,

Iβ. $a=1$, $b=4$, $c=3$, $d=2$. IIβ. $a=1$, $b=3$, $c=4$, $d=2$.

We can always choose $a=1$. If it is not 1, by a proper power of the transformation we can make it 1. Using the above values for a, b, c, d we find the set of eight equations in $\lambda_1, \ldots, \lambda_5$ incapable of solution except for the values in IIβ. Hence the P_8^3 contains no absolute constant and has the following coordinates:

$$1: 1,\ 1,\ 1,\ 1, \qquad 2: 1,\ \epsilon^{3},\ \epsilon^{4},\ \epsilon^{2}, \qquad 3: 1,\ \epsilon,\ \epsilon^{3},\ \epsilon^{4}, \qquad 4: 1,\ \epsilon^{4},\ \epsilon^{2},\ \epsilon,$$
$$5: 1,\ \epsilon^{2},\ \epsilon,\ \epsilon^{3}, \qquad 6: 1,\ 0,\ 0,\ 0, \qquad 7: 0,\ 1,\ 0,\ 0, \qquad 8: 0,\ 0,\ 1,\ 0.$$

(17) The order of association (123456) implies projectivity to the order (135)(246). Let the P_8^3 be

$$1: 1,\ 1,\ 1,\ 1, \qquad 2: \lambda,\ 1,\ x_3,\ x_4, \qquad 3: 1,\ 1,\ \omega^2,\ \omega, \qquad 4: \lambda,\ 1,\ \omega^2x_3,\ \omega x_4,$$
$$5: 1,\ 1,\ \omega,\ \omega^2, \qquad 6: \lambda,\ 1,\ \omega x_3,\ \omega^2x_4, \qquad 7: 1,\ 0,\ 0,\ 0, \qquad 8: 0,\ 1,\ 0,\ 0.$$

The sixteen equations resulting from the identity are satisfied, for the above choice of coordinates, if $\lambda=-1$, and $x_4=-\omega x_3$. Hence the set contains one absolute constant and can be written as

$$\begin{array}{llll} 1:\ 1,\ 1,\ 1,\ 1, & 2:\ -1,\ 1,\ \alpha,\ -\omega\alpha, & 3:\ 1,\ 1,\ \omega^2,\ \omega, & 4:\ -1,\ 1,\ \omega^2\alpha,\ -\omega^2\alpha, \\ 5:\ 1,\ 1,\ \omega,\ \omega^2, & 6:\ -1,\ 1,\ \omega\alpha,\ -\alpha, & 7:\ 1,\ 0,\ 0,\ 0, & 8:\ 0,\ 1,\ 0,\ 0. \end{array}$$

To construct the set choose 7, 8, H_1, H_2 as the reference points, and 1 as unit point. Let H_3 be the point on $\overline{78}$ cut out by the plane $\overline{1H_1H_2}$ and P be the fourth harmonic on $\overline{78}$ of H_3 as to 7 and 8. Then 3 and 5 are determined as those points which with 1 have $H_1H_2H_3$ as their Hessian triangle. Select 4′ in the plane $\overline{H_1H_2H_3}$ as any point on the harmonic line of $\overline{H_31}$ as to $\overline{H_3H_1}$ and $\overline{H_3H_2}$; likewise determining 5′ and 6′ as those points which with 4′ have $H_1H_2H_3$ as their Hessian triangle. Projecting from 7 upon $\overline{PH_1H_2}$ the points 4′, 5′, 6′ project into 4, 5, 6, whence the set is determined completely. By changing the sign of α, we get a set 4″, 5″, 6″ in $\overline{H_1H_2H_3}$, which projected from 8 gives 4, 5, 6 in the plane $\overline{PH_1H_2}$. The set of points lie on a cubic through 1, 2, ... , 6 and a bisecant on which lie 7 and 8. Hence the order of association (123456) is possible and the set of points contains one absolute constant.

(18) The order of association (12345678) implies projectivity to the order (1357) (2468). Let the P_8^3 be

$$\begin{array}{lll} 1:\ 1,\ 1,\ 1,\ 1, & 2:\ x_1,\ x_2,\ x_3,\ x_4, & 3:\ 1,\ i,\ -1,\ -i, \\ 4:\ x_1,\ ix_2,\ -x_3,\ -ix_4, & 5:\ 1,\ -1,\ 1,\ -1, & 6:\ x_1,\ -x_2,\ x_3,\ -x_4, \\ & 7:\ 1,\ -i,\ -1,\ i, & 8:\ x_1,\ -ix_2,\ -x_3,\ ix_4. \end{array}$$

The sixteen equations resulting from substituting these values in the identity are satisfied if we choose x_1, x_2, x_3, x_4 to satisfy $i(x_1^2-x_3^2)-(x_2^2-x_4^2)=0$, whence the set contains two absolute constants. The quadric passes through 2 and 6, and has $\overline{13}$, $\overline{57}$, $\overline{15}$, $\overline{37}$ as generators. This apparent lack of symmetry is explainable. In the order (12345678) we pick out the points 2 and 6, the numbers enclosing them are 1, 3, 5, 7, hence we use for generators besides $\overline{15}$, $\overline{37}$, the lines $\overline{13}$, $\overline{57}$. Similarly, if we isolate points 4 and 8, the numbers enclosing them are 3, 5, 1, 7; hence there is a quadric having $\overline{15}$, $\overline{37}$, $\overline{35}$, $\overline{17}$ as generators, and on 4 and 8, namely:

$$i(x_1^2-x_3^2)+(x_2^2-x_4^2)=0.$$

If we choose $x_4=1$, $x_3=b$, $x_2=a$, then $x_1=\sqrt{b^2+i(a^2-1)}$. Hence the order of association (12345678) is possible, the set contains two absolute constants, lies on an elliptic quartic, and its coordinates are given above.

(19) For the order of association (12) (34) (56) the identities are satisfied if we take points 5, 6, 7, 8 as the reference points, 4 as unit point, and 1, 2, 3 as x, y, z, respectively, by $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$, $\lambda_5=\lambda_6$, and if the following equations are satisfied,

$$\begin{aligned} x_2y_3+\ x_3y_2+x_1y_1-x_2y_2-x_1y_3-x_3y_1&=0,\\ x_2y_4+\ x_4y_2+x_1y_1-x_2y_2-x_1y_4+x_4y_1&=0,\\ x_3y_4+x_4y_3+2x_1y_1-x_1y_3-x_3y_1-x_1y_4-x_4y_1&=0. \end{aligned}$$

Solving for y in terms of x we note:

$$y_1=y_2=y_3=y_4=x_1-x_2[2x_3x_4+2x_1x_2-x_2x_4-x_2x_3-x_1x_3-x_1x_4].$$

Now the four values of y_i can not be equal, else this point would coincide with 4, neither can $x_1=x_2$, or four points would be in a plane, whence point 1 must lie on the above quadric, and then point 2 will lie on a line. Point 3 is determined by the conditions $z_1:z_2:z_3:z_4=x_1y_1:x_2y_2:x_1y_3+x_3y_1-x_1y_1:x_1y_4+x_4y_1-x_1y_1$. Hence this P_8^3 contains three absolute constants. If we choose $x_1=1$, $x_2=a$, $x_3=b$, then $x_4=\dfrac{b+ab-2a}{2b-a-1}$.

The coordinates of point 2 are

$$y_1=1+\lambda,\quad y_2=\frac{b-1}{b-a}(1-\lambda),\quad y_3=\frac{2\lambda(b-1)}{a-1},\quad y_4=\frac{2(b-1)}{2b-a-1};$$

while point 3 is

$$z_1=1+\lambda,\quad z_2=\frac{a}{b-a}[(b-1)(1-\lambda)],\quad z_3=\frac{b-1}{a-1}[2\lambda+(1+\lambda)(a-1)],$$
$$z_4=\frac{(b-1)[2+(1+\lambda)(a-1)]}{2b-a-1}.$$

Therefore the order of association (12) (34) (56) is possible; the set contains three absolute constants and its coordinates are given above.

§ 3. In this section we shall treat the P_8^3 when they are the base points of a net of quadrics. As such they are self-associated in the identical order, and if self-associated in some given order they are therefore projective in that order. The connection between the general planar quartic (genus 3) and the net of cubic curves on seven points in the plane is well known. The ∞^2 elliptic quartics on the eight base points project from one of those points into a net of cubics on seven points in the plane. Hence an intimate relation exists between the eight base points and the planar quartic. If now the P_8^3 is self-associated in some order and thereby projective in that order, we get a birational transformation of the planar quartic into itself, which is, moreover, a collineation. The types of self-projective quartics have been tabulated by Wiman, and in

this section we shall, where possible, connect each type with an order of self-association. Since the planar quartic is not of genus 3, if two points coincide, three lie on a line, or four be in a plane,* no orders of self-association will enter here which were of type E and which were not discussed in the preceding section.

Associated with the planar quartic are its thirty-six systems of contact cubics. If a collineation permutes some of these systems leaving at least one system fixed, there will be a self-association corresponding to it of the P_8^3. If, however, all of these thirty-six systems are permuted under the collineation, then we can not connect an order of self-association with the quartic. The twenty-eight double-tangents of the quartic shall be designated by the twenty-eight symbols [12], ..., [78], thus furnishing at a glance the number of double-tangents fixed under an order of association.

The ten types of self-projective quartics are:

1°	$x_3^4+x_3^2f_2(x_1, x_2)+f_4(x_1, x_2)=0,$	6°	$x_3^3x_1+x_1^4+x_1^2x_2^2+x_2^4=0,$
2°	$x_3^3f_1(x_1, x_2)+f_4(x_1, x_2)=0,$	7°	$x_3^3x_2+x_2^3x_1+x_1^3x_3=0,$
3°	$ax_3^2x_2^2+bx_3x_2x_1^2+x_3^3x_1+x_2^3x_1+x_1^4=0,$†	8°	$x_3^4+x_1^3x_2+x_1x_2^3=0,$
4°	$x_3^4+f_4(x_1, x_2)=0,$	9°	$x_3^3x_1+x_1^3x_2+x_2^4=0,$
5°	$x_3^4+ax_3^2x_1x_2+x_1^4+bx_1^2x_2^2+x_2^4=0,$†	10°	$x_3^4+x_2^4+x_1x_2^3=0.$

Quartic 1° has four fixed double-tangents. This excludes the orders of association (12) and (12)(34) as they both have more fixed lines; leaving the orders (12)(34)(56) and (12)(34)(56)(78) to be considered. Apply theorem (5) to the order (12)(34)(56). The order of association (12)(34) on the six projected points in a plane requires (a) the six points to lie on a conic, or (b) four on a line, or (c) six on a line.‡ The last two conditions would require the P_8^3 to have four points at least in a plane, while (a) makes the points lie on a quadric cone. If this happens the planar quartic associated with the P_8^3 has a double-point and is no longer of genus 3. Hence, if we can connect any order of association with quartic 1°, it must be the order (12)(34)(56)(78). This we can do, and the conditions on the elliptic parameters of the points are $u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=p/4\,(p=\text{period})$. This set contains four absolute constants; the modulus is not an absolute constant, and this necessarily is the number of constants in the equation 1°.

* This excludes the P_8^3 which form two desmic tetrahedra, which set is unaltered by a G_{192}. Of course this P_8^3 is of type B, and the planar quartic associated with it is the complete quadrilateral.

† Types given by Wiman are $x_3^2x_2^2+x_3x_2x_1^2+x_3^3x_1+x_2^3x_1+x_1^4=0$, and $x_3^4+x_3^2x_1x_2+x_1^4+x_1^2x_2^2+x_2^4=0$. The most general quartics of these types contain two constants, and are as given above.

‡ C., p. 161.

Quartic 2° is invariant under a perspective G_3 and has $x_3=0$ as a fixed undulation tangent. The group associated with quartic 3° is a non-perspective G_3 with $x_1=0$ as a fixed double-tangent. Of the two orders of association of period 3, we discard (123), since it has too many fixed lines, and study (123)(456). The P_8^3 is associated and projective in the order (123)(456). The identity can be satisfied if the set has the following coordinates:

$$1:\ 1,\ 1,\ 1,\ 1,\qquad 2:\ 1,\ 1,\ \omega^2,\ \omega,\qquad 3:\ 1,\ 1,\ \omega,\ \omega^2,\qquad 4:\ ab,\ 1,\ a,\ b,$$
$$5:\ ab,\ 1,\ \omega^2 a,\ \omega b,\qquad 6:\ ab,\ 1,\ \omega a,\ \omega^2 b,\qquad 7:\ 1,\ 0,\ 0,\ 0,\qquad 8:\ 0,\ 1,\ 0,\ 0.$$

The set contains two absolute constants, and lies on a cubic curve and its bisecant. Given the curve and points 1, 2, 3 on it, they determine a cyclic transformation of period 3. Choose 4 as any point on the curve, then 5 and 6 are determined as those points which with 4 form a cyclic set under the transformation. The two fixed points of the transformation are then known. Draw their join, and 7 and 8 are a pair of points on this line harmonic to the Weddle points on the line. Thus the set can be constructed.

Both quartics 2° and 3° contain two constants and have one tangent line fixed, but quartic 3° is invariant under a dihedral $G_{2,3}$. Whether or not the P_8^3 is invariant under a $G_{2,3}$ will decide the question as to with which quadric shall be connected the order of association (123)(456). Since we have just seen that the order of association is possible, one at least of the thirty-six systems of contact cubics is fixed. The quartic, being invariant under a G_3, must have two other systems of contact cubics fixed, and we can represent them by $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$. If the P_8^3 is invariant under a $G_{2,3}$ the transformation of period 2 can not be a harmonic perspectivity in a point and plane—else four points in a plane—and is consequently a harmonic perspectivity in two fixed lines. Moreover, it must be of the type $(ab)(cd)(ef)(gh)$ for the P_8^3, if projective in an order of period 2, is also self-associated in that order, and we saw that the above-mentioned type of period 2 is the only one that exists.

This transformation must send (123)(456) into its inverse, leave $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$ unaltered; consequently is (14)(26)(35)(78), or (15)(24)(36)(78), or (16)(25)(34)(78). But the P_8^3 is not invariant under any one of these three transformations. Hence we conclude that the order of association (123)(456) is to be connected with quartic 2° and none can be connected with quartic 3°.

Quartic 4° has four fixed undulation tangents, while quartic 5° has none. The orders of association (1234), (1234)(56), and (1234)(56)(78) are excluded since they do not have the same number of fixed lines. The remain-

ing order of period 4 has no fixed lines, hence we connect (1234) (5678) with quartic 5°. The set of points is therefore projective and self-associated in the order (1234) (5678). Let them have the coordinates:

$$\begin{array}{llll} 1:\ 1,\ 1,\ 1,\ 1, & 2:\ 1,\ i,\ -1,\ -i, & 3:\ 1,\ -1,\ 1,\ -1, \\ 4:\ 1,\ -i,\ -1,\ i, & 5:\ x_1,\ x_2,\ x_3,\ x_4, & 6:\ x_1,\ ix_2,\ -x_3,\ -ix_4, \\ & 7:\ x_1,\ -x_2,\ x_3,\ -x_4, & 8:\ x_1,\ -ix_2,\ -x_3,\ ix_4, \end{array}$$

The sixteen equations resulting from the identity are reducible to six in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$, namely:

$$\begin{array}{ll} (x_1^2-\ x_3^2)(\lambda_5+\lambda_6+\lambda_7+\lambda_8)=0, & (x_2^2\ -x_1x_3)(\lambda_5-\lambda_6+\lambda_7-\lambda_8)=0, \\ (x_1^2-x_2x_4)(\lambda_5+\lambda_6+\lambda_7+\lambda_8)=0, & (x_1x_4-x_2x_3)(\lambda_5-i\lambda_6-\lambda_7+i\lambda_8)=0, \\ (x_2^2-\ x_4^2)(\lambda_5-\lambda_6+\lambda_7-\lambda_8)=0, & (x_1x_2-x_3x_4)(\lambda_5+i\lambda_6-\lambda_7-i\lambda_8)=0. \end{array}$$

For these equations to be consistent, one at least of the multipliers must vanish. If $x_1^2-x_3^2=x_1^2-x_2x_4=0$, let $x_1=1$, $x_2=a$, then $x_3=\pm 1$, $x_4=1/a$. Using $x_3=1$, points 1, 3, 5, 7 lie in a plane, while if $x_3=-1$, points 1, 3, 6, 8 lie in a plane, so the above hypothesis is impossible. A similar argument shows that if $x_2^2-x_4^2=x_2^2-x_1x_3=0$, four points will lie in a plane, so this assumption is likewise untenable.

If $x_1x_4-x_2x_3=0$ by letting $x_1=1$, $x_2=a$, $x_3=b$, then $x_4=ab$ and the P_8^3 is non-degenerate, lying on a quartic curve. Its coordinates are:

$$\begin{array}{llll} 1:1,\ 1,\ 1,\ 1, & 2:1,\ i,\ -1,\ -i, & 3:1,\ -1,\ 1,\ -1, & 4:1,\ -i,\ -1,\ i, \\ 5:1,\ a,\ b,\ ab, & 6:1,\ ia,\ -b,\ -iab, & 7:1,\ -a,\ b,\ -ab, & 8:1,\ -ia,\ -b,\ iab. \end{array}$$

The four lines $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ are generators of a quadric on the curve. The points where these lines cut the line $\overline{e_2e_4}$ of the reference tetrahedron are, respectively, 0, 1, 0, 1; 0, 1, 0, -1; 0, 1, 0, b; 0, 1, 0, $-b$. Calling them, respectively, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the double ratio on $\overline{e_2e_4}$ of $\{\alpha_1\alpha_2\alpha_3\alpha_4\}$ is $\left(\frac{1-b}{1+b}\right)^2$.

The cross generators of the quadric through the points 1, 3, 2, 4 cut the side $\overline{e_3e_4}$ of the reference tetrahedron in four points whose double ratio is -1. Hence these cross generators are harmonic. The set contains two absolute constants.

If we assume $x_1x_2-x_3x_4=0$, we get a set of points projectively equivalent to the set just discussed, giving no new types. If we assume that more than one of the multipliers of the equations in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ vanish, then either two points coincide or four points lie in a plane. Hence the order of association (1234) (5678) is connected with quartic 5°, P_8^3 lies on an elliptic quartic and contains two absolute constants.

To quartic 4° we can connect no order of self-association.

Quartic 6° is invariant under a cyclic G_6 whose square $(1, 1, j)$ is connected with the order of association (123)(456) and whose cube $(1, 1, -1)$ with the order (12)(34)(56)(78). In examining the orders of association of period 6 we note that (123)(45) and (123)(45)(67) have not the right number of fixed lines, one. The cube of the order (123456) contains only three cycles of two numbers, while the cube of (123)(456)(78) contains one cycle of two numbers, so neither can be used. The remaining order (123456)(78) is of type E, hence the quartic would not be the general one. Therefore we can not connect an order of self-association with the quartic 6°.

Quartic 7° is the well-known quartic of Klein,* invariant under a G_{168}. With it we connect the order of association (1234567) if it exists. Let us then, with Klein, take the coordinates of the set to be:

$$\begin{array}{lll} 1:\ 1,\quad 2,\quad 2,\quad 2, & 2:\ 1,\ 2\epsilon^{-1},\ 2\epsilon^{-2},\ 2\epsilon^{-4}, & 3:\ 1,\ 2\epsilon^{-2},\ 2\epsilon^{-4},\ 2\epsilon^{-1}, \\ 4:\ 1,\ 2\epsilon^{-3},\ 2\epsilon^{-6},\ 2\epsilon^{-5}, & 5:\ 1,\ 2\epsilon^{-4},\ 2\epsilon^{-1},\ 2\epsilon^{-2}, & 6:\ 1,\ 2\epsilon^{-5},\ 2\epsilon^{-3},\ 2\epsilon^{-6}, \\ & 7:\ 1,\ 2\epsilon^{-6},\ 2\epsilon^{-5},\ 2\epsilon^{-3}, & 8:\ 1,\quad 0,\quad 0,\quad 0, \end{array}$$

where $\epsilon^7=1$.

The sixteen equations resulting from substituting these values in the identity are satisfied by choosing $\lambda_1=\lambda_2=\ldots.=\lambda_7=1$. Thus the set exists, contains no absolute constant and is connected with quartic 7°. With the above coordinates the transformations S and T of Klein are, respectively, (1234567) and (18)(27)(34)(56). Not only is this set self-associated in the identical order and the order (1234567) but in one hundred and sixty-six others. By using S and T we can easily find them and learn they are of four types: forty-eight of type (178)(246), fifty-six of type (1358)(2674), twenty-one of type (18)(27)(34)(56), forty-two of type (1234567), which, with the identity, make the one hundred and sixty-eight.

The multipliers of the group leaving 8° unaltered are $1, -1, \frac{1+i}{2}$. The square of this is $1, 1, i$; the group leaving 4° unaltered. There is no order of self-association connected with 4° and, consequently, none can be connected with quartic 8°.

No order of association of period nine appears among those of our list, so we have none to connect with quartic 9°.

Quartic 10° is invariant under a G_{12}. The only transformation of period 12 is (1234)(567) which we saw was of type E. Therefore quartic 10° cannot be connected with an order of association.

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* Klein, "Elliptischen Modulfunkturiem," Bd. I, pp. 701, 724–725.

Hence only four of the ten types of self-projective quartics can be connected with an order of self-association of the P_8^3, namely:

1° with (12) (34) (56) (78) 5° with (1234) (5678),
2° with (123) (456), 7° with (1234567).

§ 4. If the P_8^3 lies on a rational cubic, it is self-associated in the identical order. Excluding all of type E we find that the P_8^3 on a rational cubic is projective and hence self-associated in the following orders: (12) (34) (56), (12) (34) (56) (78), (123) (456), (1234) (5678), (123456), (1234567), and (12345678). The binary octavics giving the parameters of the P_8^3 for each order of self-association are, respectively:

$$(x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(x_1^2+\beta x_2^2)x_1x_2=0, \qquad (x_1^4+x_2^4)(x_1^4+\alpha x_2^4)=0,$$
$$(x_1^2+x_2^2)(x_1^2+\alpha x_1^2)(x_1^2+\beta x_2^2)(x_1^2+\gamma x_2^2)=0, \qquad (x_1^6+x_2^6)x_1x_2=0,$$
$$(x_1^3+x_2^3)(x_1^3+\alpha x_2^3)x_1x_2=0, \qquad (x_1^7+x_2^7)x_1=0, \quad (x_1^8+x_2^8)=0.$$

The number of absolute constants for each P_8^3 is obvious from the above octavics.

In § 2 and § 3 are given the possible types of self-associated sets. A problem that suggests itself for the future is to determine those P_8^3's which are self-associated in more than one order, and the groups connected with them. This was done for the order (1234567) in § 3, and the one hundred and sixty-eight ways of self-association discussed. We shall do the same now for the P_8^3 on a rational cubic.

In studying the dihedral groups connected with the above P_8^3 we need look for a $G_{2,n}$ only for $n<5$ if n is odd. If $n \geqq 5$, $2n>8$, and with the odd integer n we could use only n again, which likewise makes a number greater than 8. Hence for n odd $n \leqq 3$. Similarly for n even $n \leqq 8$. Therefore the possible G_{2n}'s must be found where $n=2, 3, 4, 6, 8$. In the accompanying table are given the numbers for each value of n.

2	3	4	6	8
2	2	2	2	2
2	3	4	6	8
2	3	4	6	8
4	6	8	12	16

The $G_{2,2}$ has two types	$(x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(\alpha x_1^2+x_2^2)x_1x_2$,
	$(x_1^4+\alpha x_1^2x_2^2+x_2^4)(x_1^4+\beta x_1^2x_2^2+x_2^4)$.
The $G_{2,3}$ has two types	$(x_1^6-x_2^6)x_1x_2$, $(x_1^6+\alpha x_1^3x_2^3+x_2^6)x_1x_2$.
The $G_{2,4}$ has two types	$(x_1^8-x_2^8)$, $(x_1^8+\alpha x_1^4x_2^4+x_2^8)$.
The $G_{2,6}$ has one type	$(x_1^6+x_2^6)x_1x_2$.
The $G_{2,8}$ has one type	$x_1^8+x_2^8$.
The tetrahedral G_4 is	$x_1^8-14x_1^4x_2^4+x_2^8$.

Conclusion.

The general P_8^3 on an elliptic quartic can be self-associated in the following orders: (12), (12)(34), (12)(34)(56), (12)(34)(56)(78), (123)(456)(78), (1234)(5678), (12345)(67), (123456) and (12345678).

The P_8^3, which is the base points of a net of quadrics, can be self-associated in the following orders, besides the identity: (12)(34)(56)(78), (123)(456), (1234)(5678) and (1234567). To each of these we have connected a particular planar quartic (genus 3), which is self-projective.

The P_8^3 on a rational cubic can be self-associated in the following orders: (12)(34)(56), (12)(34)(56)(78), (123)(456), (1234)(5678), (123456), (1234567) and (12345678). The groups connected with sets are also given. The discussion throughout the paper was restricted to sets of points, of which no two coincide, no three lie on a line, no four in a plane.

Associate Minimal Surfaces.

BY JAMES K. WHITTEMORE.

It is a well-known fact, first proved by Schwarz,* that corresponding points of a family of associate minimal surfaces, corresponding in a sense presently to be explained, lie on an ellipse. In Part I of this paper we find the locus of the extremities of these ellipses, which we call Schwarz's ellipses, then find the envelope of a family of associate minimal surfaces and prove that the latter coincides with part of the locus named for real minimal surfaces applicable to surfaces of revolution, with certain exceptions; in Part II it is shown that this coincidence, with coincidence of corresponding points, occurs only when the minimal surfaces, supposed real, are applicable to a surface of revolution.

The Enneper-Weierstrass equations of a minimal surface S are

$$\left.\begin{aligned} x&=\frac{1}{2}\int(1-u^2)F(u)\,du+\frac{1}{2}\int(1-v^2)\phi(v)\,dv=U_1+V_1,\\ y&=\frac{i}{2}\int(1+u^2)F(u)\,du-\frac{i}{2}\int(1+v^2)\phi(v)\,dv=U_2+V_2,\\ z&=\int uF(u)\,du+\int v\phi(v)\,dv=U_3+V_3. \end{aligned}\right\} \quad (1)$$

The parameters u, v are the parameters of the minimal curves of S. When S is real F and ϕ are conjugate functions, and for a real point with a real tangent plane u, v have conjugate values.† The *adjoint* surface has the equations

$$x_1=i(U_1-V_1),\quad y_1=i(U_2-V_2),\quad z_1=i(U_3-V_3).$$

The equations of the associate minimal surface S_a are

$$\left.\begin{aligned} x_a&=U_1e^{ia}+V_1e^{-ia}=x\cos\alpha+x_1\sin\alpha,\\ y_a&=U_2e^{ia}+V_2e^{-ia}=y\cos\alpha+y_1\sin\alpha,\\ z_a&=U_3e^{ia}+V_3e^{-ia}=z\cos\alpha+z_1\sin\alpha. \end{aligned}\right\} \quad (2)$$

Values of α differing by $\pi/2$, substituted in (2), give adjoint surfaces; corresponding points of associate minimal surfaces are given by (2) when u, v are

* H. A. Schwarz, "Miscellen aus dem Gebiete der Minimalflächen," *Journal de Crelle*, Vol. LXXX (1875).

† Eisenhart, "Differential Geometry," p. 256.

fixed and α varies. The locus of $x_\alpha, y_\alpha, z_\alpha$ for fixed u, v is Schwarz's ellipse. Points of the ellipse on adjoint surfaces are the extremities of conjugate diameters.*

I.

§ 1. **The Locus L.** To find the extremities of the principal axes of the ellipse (2) we determine a point (α) such that the tangent at this point is perpendicular to the line joining it with the center. This condition gives

$$\Sigma(x \cos\alpha + x_1 \sin\alpha)(-x\sin\alpha + x_1\cos\alpha) = 0,$$

from which

$$\tan 2\alpha = \frac{2\Sigma x x_1}{\Sigma(x^2 - x_1^2)} = i\,\frac{\Sigma(U_1^2 - V_1^2)}{\Sigma(U_1^2 + V_1^2)}.$$

The last equation determines two values of α, differing by $\pi/2$, unless $\Sigma U_1^2 = \Sigma V_1^2 = 0$. The vanishing of these two sums is the condition that the ellipse be a circle. It may easily be proved that if Schwarz's ellipse is a circle for all points of a real minimal surface the latter is a plane. We find

$$\cos 2\alpha = \pm\frac{\Sigma(U_1^2 + V_1^2)}{2\sqrt{\Sigma U_1^2 \Sigma V_1^2}}.$$

Choosing the upper sign,

$$\cos\alpha = \pm\frac{1}{2}\,\frac{\sqrt{\Sigma U_1^2} + \sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2 \Sigma V_1^2}}, \quad \sin\alpha = \pm\frac{i}{2}\,\frac{\sqrt{\Sigma U_1^2} - \sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2 \Sigma V_1^2}}.$$

Choosing again the upper signs, we have for one vertex, from (2),

$$x = U_1\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}} + V_1\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}}, \tag{3}$$

with similar expressions for y and z. The two radicals in (3) are reciprocals, and conjugate for conjugate u, v. The other combinations of signs in the preceding equations give the other vertices of the ellipse; it appears that the four vertices are given by (3) by the four different determinations of the first radical. The locus L is the locus of the four vertices of Schwarz's ellipses, and its equations, in the parameters u, v, are given by (3) with the similar equations for y and z. The locus consists evidently of four nappes, symmetrical in pairs with respect to the origin; these symmetrical pairs we call L_1, L_2 and L_3, L_4. The squares of the semi-axes of the ellipse are given by $2(\Sigma U_1 V_1 \pm \sqrt{\Sigma U_1^2 \Sigma V_1^2})$.

* Scheffers, "Einführung in die Theorie der Flächen," 2d ed., p. 362.

Comparing equations (2) and (3) it is evident that the curve $\sqrt[4]{\Sigma V_1^2/\Sigma U_1^2}=e^{i\alpha}$ is common to the two surfaces S_α and L_1; similarly, curves given by

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-e^{i\alpha},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-ie^{i\alpha},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=ie^{i\alpha},$$

are common, respectively, to S_α and L_2, S_α and L_3, S_α and L_4; it is to be noted that the first of the last three equations, for example, is also the equation of a curve common to $S_{\alpha+\pi}$ and L_1.

It may be proved that a part of the locus L coincides with one of the associate minimal surfaces only if the latter are plane.

§ 2. **The Envelope of S_α.** If we substitute in (2) the value of α in terms of u, v taken from the equation $|\partial x/\partial u\ \partial y/\partial v\ \partial z/\partial \alpha|=0$, we should expect to find both the envelope and the locus of singular points of the surfaces S_α, but it appears that only the envelope results, for the equation is an identity in α at the singular points. For the envelope it gives

$$e^{i\alpha}=\pm\sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}}.$$

Evidently the envelope consists of two nappes symmetrical with respect to the origin.

§ 3. **Minimal Surfaces Applicable to Surfaces of Revolution.** We apply the results of the preceding sections to families of real associate minimal surfaces applicable to surfaces of revolution.* All such surfaces are given by (1) where,

$$F(u)=cu^{m-2},\quad \phi(v)=\bar{c}v^{m-2},$$

m being a real constant, c and $\bar{c}$ conjugate constants. We call such surfaces B surfaces, as they were discovered by E. Bour, in particular a surface for any special value of m, not zero, B_m. It has been proved that all the surfaces associate to B_m are congruent, so that B_m is defined except for a homothetic transformation, and we may without restriction suppose c real; the associates of B_m are obtained by rotating the latter about the Z-axis; B_m and B_{-m} are congruent, so that we may suppose m positive; the curves of the surface, v/u constant, are geodesics and correspond to the meridians of the applicable surface of revolution; the curves, uv constant, are curves of constant total curvature, and correspond to the parallels of the surface of revolution. It is easily shown that the curves of B_m

$$\left(\frac{v}{u}\right)^m=1,\quad \left(\frac{v}{u}\right)^m=-1,$$

* We have given an account of these surfaces and of the literature concerning them in a paper published in the *Annals of Mathematics*, Second Series, Vol. XIX, No. 1, September, 1917.

are respectively lines of curvature and asymptotic lines, and that the surface, when the constants of integration are taken as zero, is cut by the xy-plane in the latter. The value $m=0$ gives the minimal helicoids including the catenoid and the right helicoid; $m=2$ gives Enneper's surface.

Excluding $m=0, 1$, and taking all constants of integration as zero, we find

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=\sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}}=\left(\frac{v}{u}\right)^{m/2}.$$

It follows that, for B_m, L_1 and L_2 coincide with the envelope of the associate surfaces. It is readily shown that L_1 is a surface of revolution whose axis is the Z-axis, and that L_3 is the xy-plane λ. If $m>1$, L_1 and L_2 form the locus of the extremities of the minor axes of the Schwarz ellipses, while L_3 and L_4, coinciding with the xy-plane, contain the major axes; if $m<1$, the situation is reversed. All ellipses corresponding to points of a curve of constant total curvature of B_m are equal. The curve previously mentioned, common to B_m and L_1 is $(v/u)^{m/2}=1$, which gives $y/x=\tan(2k\pi/m)$ where k is any integer; for B_m and L_2, $(v/u)^{m/2}=-1$ giving $y/x=\tan[(2k+1)\pi/m]$. The curves on L_1 are congruent plane curves, being meridians of the surface of revolution; they are lines of curvature and also geodesics of B_m, which is tangent to L_1 along each curve. The plane of each curve is a plane of symmetry of B_m. Similar statements apply to B_m and L_2. The curves common to B_m and L_3, B_m and L_4 are given respectively by

$$\left(\frac{v}{u}\right)^{m/2}=-i;\quad \frac{y}{x}=-\cot\frac{(2k+\frac{1}{2})\pi}{m},\quad z=0,$$

$$\left(\frac{v}{u}\right)^{m/2}=\ i;\quad \frac{y}{x}=-\cot\frac{(2k-\frac{1}{2})\pi}{m},\quad z=0,$$

and are straight lines in the xy-plane. Each line is a line of symmetry of B_m.

From the previous paragraph follow several theorems first proved by Ribaucour. A surface B_m, $|m|\neq 1$, has a number of congruent plane geodesic lines of curvature lying in planes which contain the Z-axis; half or all of these, depending on the value of m, are obtained by rotating one of them through successive angles $2\pi/m$, and an equal number are obtained by rotating about the Z-axis the symmetry of one of these curves with respect to the xy-plane through the angle π/m, then through successive angles $2\pi/m$; straight asymptotic lines of the surface, lying in the xy-plane, bisect the angles of the planes of the lines of curvature named above; if $m=p/q$, where p and q are integers with no common factor, and $q=1$ if m is an integer, the number of these plane lines of curvature is p and is equal to the number of straight

asymptotic lines referred to; if m is irrational the number of each of these sorts of lines is infinite. Ribaucour's results are not so fully stated nor are they completely proved.

II.

§ 4. **Formulation of the Converse Problem.** The remainder of this paper is devoted to the investigation of the question: For what real minimal surfaces does the envelope of the associate surfaces coincide with part of the locus L, with coincidence of points given by the same u, v? We prove that the only surfaces having this property are the plane and B_m, $|m| \neq 1$, so that this coincidence is a characteristic property of real minimal surfaces applicable to surfaces of revolution. It must, however, be remarked that it does not hold for B_1 or for the minimal helicoids, $m=0$.

For the required coincidence we must have, for all u, v, either $U_1/V_1 = U_2/V_2 = U_3/V_3$ or

$$\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3} = \pm\frac{\sqrt{\Sigma V_1^2}}{\sqrt{\Sigma U_1^2}}. \tag{4}$$

where we may assume without restriction that the radicals in (4) are conjugate for conjugate u, v. If the U's are proportional to the V's all six of these functions are constants so that the surface (1) is a point; this condition is also included in (4) so that the latter is the necessary and sufficient condition for the required coincidence.

Choosing first the upper sign in (4) we rewrite it in the form

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}}v+\frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}} = \frac{V_1-iV_2+vV_3}{\sqrt{\Sigma V_1^2}}u+\frac{vV_1+ivV_2-V_3}{\sqrt{\Sigma V_1^2}}. \tag{4'}$$

Since (4′) is an identity in u, v the first number is linear in u, and as there can be no cancellation in its two terms we have

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}} = au+b, \quad \frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}} = c+du, \tag{5}$$

where a, b, c, d are constants. From (4′) it follows that a and c are real, b and d are conjugate. If we take the lower sign in (4) we are led again to (5), but in this case a and c are pure imaginaries, b and $-d$ are conjugate.

The following discussion consists in the study of (5), to which we add the equations, implied in (1), $\Sigma U_1'^2=0$, and

$$U_1'+iU_2'+uU_3'=0. \tag{6}$$

Before undertaking the solution of these equations for U_1, U_2, U_3 we transform (5), writing

$$\sqrt{\Sigma U_1^2}=\rho, \quad U_1=\rho\lambda, \quad U_2=\rho\mu, \quad U_1=\rho\nu,$$

where, evidently,

$$\Sigma\lambda^2=\lambda^2+\mu^2+\nu^2=1, \quad \Sigma\lambda\lambda'=0.$$

From $\Sigma U_1'^2=0$,

$$\rho^2\Sigma\lambda'^2+\rho'^2=0, \quad \rho=e^{\pm i\int\sqrt{\Sigma\lambda'^2}du}.$$

Equations (5) may now be written

$$\lambda+i\mu=u(a-\nu)+b, \quad \lambda-i\mu=\frac{1}{u}(c+\nu)+d. \tag{5'}$$

Our problem may now be regarded as consisting of the solution of (5′) with $\Sigma\lambda^2=1$ for λ, μ, ν, then the determination of ρ and hence U_1, U_2, U_3, to be followed by applying (6) as a necessary condition.

We multiply together equations (5′) and replace $\Sigma\lambda^2$ by unity, finding

$$\nu=\frac{1-ac-bd-adu-bc/u}{a-c-du+b/u}.$$

Differentiating (5′) and multiplying together the resulting equations,

$$\Sigma\lambda'^2=\nu'(a+c)/u-[ac+\nu(a-c)-\nu^2]/u^2.$$

§ 5. **Solution for $b=d=0$, $a\neq c$.** For B_m we find $b=d=0$; it is therefore natural to attempt first the solution of the equations of § 4 on the assumption, $b=d=0$. We suppose further $a\neq c$. We find

$$\nu=\frac{1-ac}{a-c}, \quad \Sigma\lambda'^2=\frac{(1-a^2)(1-c^2)}{u^2(a-c)^2},$$

from which

$$\rho=Au^{\pm i\frac{\sqrt{(1-a^2)(1-c^2)}}{a-c}},$$

where A is an arbitrary constant. The values of λ and μ are found from (5′), and from them and the given values of ν and ρ we obtain U_1, U_2, U_3. We apply to the latter the condition (6); it appears that $a^2=1$ or $a^2c^2=1$. If a^2 or $ac=1$ the surface (1) is plane. It remains to consider only $ac=-1$. Replacing c by $-1/a$ we have two sets of values for the U's corresponding to the double sign in ρ; one of these satisfies (6) only if $a^2=1$, and may be discarded; the other satisfies (6) identically, and is

$$U_1=-\frac{Aa}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}+\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_2=\frac{Aai}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}-\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_3=\frac{2Aa}{1+a^2}u^{\frac{1-a^2}{1+a^2}}.$$

These give the equations for B_m if we set

$$a=\pm i\sqrt{\frac{m-1}{m+1}},\quad A=\mp\frac{ci}{m\sqrt{m^2-1}}.$$

The first of the last two equations gives a real finite value for m unless $a=\pm i$, but such values of a are inadmissible, for both give, from $ac=-1$, $a=c$, a case excluded from our present consideration. The absolute value of m is greater or less than one as a is pure imaginary or real, so that these two cases of B_m, which show many geometrical differences, are also distinguished by the nature of a. It is interesting to observe that $m=0, 1$ are given by $a^2=0, 1$, respectively. These are the values of m excluded in § 3; the value $a^2=0$ is now excluded since $ac=-1$; $a^2=1$ gives for (1) a plane.

§ 6. **Solution for $b=d=0$, $a=c$.** We prove that when $b=d=0$ and $a=c$ the surface (1) is plane. Replacing c by a and writing $b=d=0$ in (5′) we have

$$\lambda+i\mu=(a-\nu)u,\quad \lambda-i\mu=(a+\nu)/u,$$

from which $\Sigma\lambda^2=a^2=1$. We may suppose $a=1$, for changing the sign of a merely changes the signs of λ, μ, ν and hence the signs of x, y, z in (1). It is not now possible to determine ν as in § 4. We find

$$\Sigma\lambda'^2=(\nu^2+2u\nu'-1)/u^2,$$

and substitute this value in the expression for ρ of § 4. Further,

$$2U_1=[1/u+u+\nu(1/u-u)]\rho,\quad 2U_2=[1/u-u+\nu(1/u+u)]\rho i,\quad U_3=\nu\rho;$$

substituting these and the value of ρ in (6),

$$\nu-1=\pm i\sqrt{\nu^2+2u\nu'-1},$$

from which $\nu=cu/(cu-1)$ and $\rho=c'(cu-1)/cu$, where c and c' are constants. Then $U_3=c'$ and (1) is the plane, $z=$constant.

§ 7. **Reduction of the General Case to $b=d=0$.** We now consider (5′) with no hypothesis concerning a, b, c, d. There are two cases: (I) a and c real, b and d conjugate, (II) a and c pure imaginary, b and $-d$ conjugate.

Since equations (5′) involve only u we may regard their transformation as that of the minimal curve,

$$x=U_1,\quad y=U_2,\quad z=U_3.$$

We prove in this section that this curve may be obtained by solving the equations of the preceding section after making a suitable linear transformation of u and a certain rotation of the coordinate axes to which the curve is referred. In the following section we show that the rotation is real, thereby proving that the surface found is unchanged by the transformation.

For any point of the curve the value of ρ is unaffected by a change of variable or by a rotation of the axes. Supposing the new equations of the curve to be

$$x_1=\rho\lambda_1, \quad y_1=\rho\mu_1, \quad z_1=\rho\nu_1,$$

we have $\lambda_1=\alpha_1\lambda+\beta_1\mu+\gamma_1\nu$, with similar equations for μ_1 and ν_1, where the coefficients are the terms of an orthogonal substitution. We write

$$u=\frac{\alpha u_1+\beta}{\gamma u_1+\delta},$$

and prove that by a suitable choice of this substitution and of the rotation (5′) becomes

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1, \tag{7}$$

where a_1 and c_1 are both real or both pure imaginary, and $a_1c_1=-1$. Substituting for u in (5′) and clearing of fractions,

$$u_1[\gamma(\lambda+i\mu)+\alpha\nu]+\delta(\lambda+i\mu)+\beta\nu=u_1(a\alpha+b\gamma)+a\beta+b\delta,$$
$$u_1[\alpha(\lambda-i\mu)-\gamma\nu]+\beta(\lambda-i\mu)-\delta\nu=u_1(c\gamma+d\alpha)+c\delta+d\beta.$$

We make two combinations of the last equations, for the first multiplying the first equation by l, the second by l', and adding; for the second using m and m' as multipliers. We require that the first members of the equations so formed be respectively the first numbers of (7). This condition gives us values for λ_1, μ_1, and two values for ν_1, linear in λ, μ, ν; equating like coefficients in the two values of ν_1,

$$l=-m\delta/\gamma, \quad l'=m\beta/\gamma, \quad m'=-m\alpha/\gamma.$$

Substituting these values in λ_1 and μ_1, then using $\Sigma\lambda_1^2=\Sigma\lambda^2=1$, we find

$$\gamma^2/m^2=(\alpha\delta-\beta\gamma)^2,$$

and choose $m=-\gamma/(\alpha\delta-\beta\gamma)$, and both pairs of multipliers are determined. A rotation has now been found so that for an arbitrary linear substitution for u the first members of the transformed (5′) have the desired form; the coefficients of the substitution may now be chosen to simplify the second members of the equations. Equations (5′) are now replaced by

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1+b_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1+d_1u_1,$$

where

$$\begin{aligned}
a_1\Delta&=\ \ a\alpha\delta+b\gamma\delta-c\beta\gamma-d\alpha\beta,\\
b_1\Delta&=\ \ a\beta\delta+b\delta^2-c\beta\delta-d\beta^2,\\
c_1\Delta&=-a\beta\gamma-b\gamma\delta+c\alpha\delta+d\alpha\beta,\\
d_1\Delta&=-a\alpha\gamma-b\gamma^2+c\alpha\gamma+d\alpha^2,\\
\Delta&=\ \ \alpha\delta-\beta\gamma.
\end{aligned}$$

From these values it appears that

$$a_1+c_1=a+c, \quad a_1c_1-b_1d_1=ac-bd.$$

We wish so to determine $\alpha, \beta, \gamma, \delta$ that $b_1=d_1=0$. This requires, since we must have $\Delta \neq 0$, that β/δ and α/γ be different roots of the quadratic, $dx^2+(c-a)x-b=0$. That the roots of this equation are distinct follows immediately since the discriminant can not vanish when (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate. Calling the roots of the quadratic x_1, x_2, writing $\alpha=\gamma x_1$, supposing $\delta=1$, as we may if $d \neq 0$, we have

$$a_1=\frac{b+ax_1-cx_2-dx_1x_2}{x_1-x_2}, \quad c_1=\frac{-b-ax_2+cx_1+dx_1x_2}{x_1-x_2}, \quad u=\frac{\gamma x_1u_1+x_2}{\gamma u_1+1}.$$

The values of a_1 and c_1 are independent of γ, so that the surface satisfying the requirement of our problem is applicable to itself in an infinite number of ways. It will appear in the next section that the surface is unchanged by the transformation when γ depends on an arbitrary real parameter. It will also appear that the new Z-axis is independent of γ.

If we apply (6) to the general equations (5′) we find, after rather tedious algebraic work, that for all surfaces other than the plane this condition becomes $ac-bd=-1$. This condition might certainly have been foreseen as necessary, for we have observed that $ac-bd$ is invariant under the transformation, and $ac=-1$ if $b=d=0$.

The values of a_1 and c_1 may also be found from the equations,

$$a_1+c_1=a+b, \quad a_1c_1=-1.$$

These give two pairs of values which exchange a_1 and c_1, and correspond to the interchange of x_1 and x_2. These two solutions give to m in B_m values differing only in sign, and therefore lead to the same surface. It may easily be proved that a_1 and c_1 are both real or both pure imaginary with a and c.

§ 8. **Reality of the Rotation.** Evidently to bring the minimal curve,

$$x=V_1, \quad y=V_2, \quad z=V_3,$$

into a corresponding reduced form, v must be transformed by a substitution conjugate to that applied to u, and the coordinate axes subjected to the rotation conjugate to that employed in § 7. To prove that the surface (1) is not changed by these two rotations we show that they are real and therefore identical. This proof may be given in two ways, both of which we indicate. The coefficients of λ, μ, ν in the expressions for λ_1, μ_1, ν_1, mentioned in § 7,

may be expressed in terms of x_1, x_2, γ. The coefficients of v_1 do not contain γ, so that the new Z-axis is the same for all reducing transformations. It may be shown directly that if (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate, and if further $\gamma=\sqrt{x_2/x_1}\,e^{i\phi}$, where ϕ is an arbitrary real number, all nine coefficients are real, hence that the surface satisfying the requirement of our problem is a surface B_m, $|m|\neq 1$. It appears also that the coordinate axes to which B_m in its standard form is referred, depend on an arbitrary real parameter, agreeing with a property of the associate surfaces mentioned in § 3. A second method of proving the rotation real consists in showing the rotation identical with its conjugate for the value of γ given above. This may easily be done if we note that $-1/x_2$, $-1/x_1$, $\sqrt{x_2/x_1}\,e^{-i\phi}$ are conjugate respectively to x_1, x_2, $\sqrt{x_2/x_1}\,e^{i\phi}$.

It is interesting to note that the linear transformation connecting u and u_1 is a real rotation of the sphere of the complex variable u for the value of γ given, and also that the transformation connecting the spheres on which u, v and u_1, v_1 are respectively the parameters of the minimal lines is a real rotation.

YALE UNIVERSITY, 1917.

On Integral Invariants.

By F. W. Reed.

The work of Poincaré* on integral invariants has been extended along the same lines by De Donder† and Goursat.‡ Lie§ has written upon a class of linear integral invariants somewhat more general than those considered by Poincaré.

In the present paper the method of infinitesimal transformations developed by Lie|| is applied to those types defined and discussed by Poincaré. The conditions for invariants of order p are found as a system of ordinary linear differential equations, which are the same as those found by Goursat. The relation between the solutions of the systems of various orders is brought out in full. In particular it is shown how all the linear invariants depend upon the original system and how all the invariants of higher order can be constructed when these are known. An extension of Poisson's theorem is given. Further it is shown that the least action integral of dynamics is nothing other than an integral invariant of the equations of motion.

§1. *Invariants of Order p.*

Consider the multiple integral

$$I_p = \int^p \Sigma A_{\alpha_1 \ldots \alpha_p} dx_{\alpha_1} dx_{\alpha_2} \ldots dx_{\alpha_p}, \tag{a}$$

where

$$A_{\alpha_1 \ldots \alpha_p} = \pm A_{\alpha'_1 \ldots \alpha'_p},$$

the upper or lower sign being taken according as the rearrangement of the $\alpha_1 \ldots \alpha_p$ in the new order $\alpha'_1 \ldots \alpha'_p$ is obtained by an even or odd number of consecutive changes, and the Σ sign is extended to all the combinations of the

* Poincaré, *Journal de l'École Polytechnique*, 2e série, premier cahier (1895). "Les Méthodes Nouvelles de la Mécanique Céleste, Tome III; *Acta Mathematica*, 13 (1890).

† De Donder, *Rendiconti del Circolo Matematico di Palermo*, 15, 16 (1901, 1902).

‡ Goursat, *Journal de Mathématiques pures et appliquées*, 6e série, Tome IV (1908).

§ Lie, "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 49 (1897), pp. 342, 369.

|| Lie, "Theorie der Transformationsgruppen."

n letters $\alpha_1 \ldots . \alpha_n$ taken p at a time. This integral is equivalent to the integral

$$I_p = \int^p \Sigma A_{\alpha_1 \ldots . \alpha_p} \frac{\partial(x_{\alpha_1} \ldots . x_{\alpha_p})}{\partial(y_1 \ldots . y_p)} dy_1 \ldots . dy_p. \tag{b}$$

The interchange of x_{α_i} and $x_{\alpha_{i+1}}$ in any one of the functional determinants changes the sign of the corresponding term in (b), but the sign may be restored by making the same permutation of α_i and α_{i+1} in the subscript of $A_{\alpha_1 \ldots . \alpha_p}$. The same changes have the same effect upon any particular term of (a). It is sufficient, therefore, when a choice of the subscripts of $A_{\alpha_1 \ldots . \alpha_p}$ is once made to write the subscripts in $dx_{\alpha_1} \ldots . dx_{\alpha_p}$ in the *same order.*

To facilitate the writing the following symbols are introduced:

$$(1 \ldots . p-2, \lambda, p) \equiv A_{\alpha_1 \ldots . \alpha_{p-2}, \lambda, \alpha_p}, \text{ etc.}$$

$$[1, \lambda] = \frac{\partial X_{\alpha_1}}{\partial x_\lambda}, \text{etc.} \quad [\lambda, p] = \frac{\partial X_\lambda}{\partial x_{\alpha_p}}.$$

Under the transformation

$$dx_i = X_i dt, \tag{1}$$

or

$$X(f) = \frac{\partial f}{\partial x_1} X_1 + \frac{\partial f}{\partial x_2} X_2 + \ldots . + \frac{\partial f}{\partial x_n} X_n,$$

the integral (a) becomes

$$I'_p = \int^p \sum_{\alpha_1 \ldots . \alpha_p} [(1 \ldots . p) + X(1 \ldots . p) dt] [dx_{\alpha_1} + \Sigma[1, \lambda] dx_\lambda dt]$$
$$[dx_{\alpha_2} + \Sigma[2, \lambda] dx_\lambda dt] \ldots . [dx_{\alpha_p} + \Sigma[p, \lambda] dx_\lambda dt]. \tag{a'}$$

The term of zero degree in dt is I_p of (a). Equating to zero the coefficient of $dx_{\alpha_1} \ldots . dx_{\alpha_p}$ in the term of first degree in dt we have

$$\Sigma[X(1 \ldots . p) + (1 \ldots . p-1, \lambda)[\lambda, p] + (1 \ldots . p-2, \lambda, p)[\lambda, p-1]$$
$$+ \ldots . + (\lambda, 2 \ldots . p)[\lambda, 1]] = 0. \tag{2}$$

If these conditions are satisfied I_p is an integral invariant under the transformation.

THEOREM I. *If an I_1 and an I_p are known, an I_{p+1} may be written by means of the relation*

$$(1 \ldots . p+1) = (1)(2 \ldots . p+1) - (2)(1, 3 \ldots . p+1) + \ldots . \pm (p+1)(1 \ldots . p).$$

Substitute this expression for $(1 \ldots . p+1)$ in the general condition (2). The respective terms become

$$X(1 \ldots . p+1) = X((1)(2 \ldots . p+1) - (2)(1, 3 \ldots . p+1)$$
$$+ \ldots . \pm (p+1)(1 \ldots . p)),$$

$$\Sigma(1\ldots.p,\lambda)[\lambda,p+1]=\Sigma[(1)(2\ldots.p,\lambda)-(2)(1,3\ldots.p,\lambda)$$
$$+\ldots.\pm(\lambda)(1\ldots.p)][\lambda,p+1],\qquad \lambda(=)\alpha_{p+1},$$
$$\Sigma(1\ldots.p-1,\lambda,p+1)[\lambda,p]=\Sigma[(1)(2\ldots.p-1,\lambda,p+1)$$
$$-(2)(1,3\ldots.p-1,\lambda,p+1)$$
$$+\ldots.\pm(p+1)(1\ldots.p-1,\lambda)][\lambda,p],\quad \lambda(=)\alpha_p,$$

....

$$\Sigma(1,\lambda,3\ldots.p+1)[\lambda,2]=\Sigma[(1)(\lambda,3\ldots.p+1)-(\lambda)(1,3\ldots.p+1)$$
$$+\ldots.\pm(p+1)(1,\lambda,3\ldots.p)][\lambda,2],\quad \lambda(=)\alpha_2,$$
$$\Sigma(\lambda,2\ldots.p+1)[\lambda,1]=\Sigma[(\lambda)(2\ldots.p+1)-(2)(\lambda,3\ldots.p+1)$$
$$+\ldots.\pm(p+1)(\lambda,2\ldots.p)][\lambda,1],\quad \lambda(=)\alpha_1,$$

where $\lambda(=)\alpha_i$ means λ different for all the $\alpha_1\ldots.\alpha_{p+1}$ except α_i. By adding to the second equation

$$0=(1)(2\ldots.p,1)[1,p+1]-(2)(1,3\ldots.p,2)[2,p+1]$$
$$+\ldots.\pm(p)(1\ldots.p-1,p)[p,p+1]$$
$$\pm[(1)(1\ldots.p)[1,p+1]+(2)(1\ldots.p)[2,p+1]$$
$$+\ldots.+(p)(1\ldots.p)[p,p+1]],$$

the gaps are filled and we have

$$\Sigma(1\ldots.p,\lambda)[\lambda,p+1]=\Sigma[(1)(2\ldots.p,\lambda)-(2)(1,3\ldots.p,\lambda)$$
$$+\ldots.\pm(\lambda)(1\ldots.p)[\lambda,p+1]],\quad \lambda(=)\alpha_{p+1},$$

where λ takes all values consistent with the properties of $(1\ldots.p)$ previously indicated. The remaining equations may be reduced in a similar manner. Adding now the first terms in the right members of the equations thus reduced the sum is

$$(1)\Sigma[X(2\ldots.p+1)+(2\ldots.p,\lambda)[\lambda,p+1]+\ldots.+(\lambda,3\ldots.p+1)[\lambda,2]]$$
$$+(2\ldots.p+1)\Sigma[X(1)+(\lambda)[\lambda,1]].$$

This expression is zero when the (i) and the $(1\ldots.p)$ are coefficients in an I_1 and an I_p, respectively. The sum of the second terms vanish in a similar manner, etc. Thus the conditions for an I_{p+1} are fulfilled.

THEOREM II. *The product of p linearly independent invariants I_1 is an I_p.*

Suppose p linear invariants are known

$$I_1^{(i)}=\int L_i\qquad(i=1,2\ldots.p),$$

where

$$L_1=A_1^1dx_1+\ldots.+A_n^1dx_n=(1)^1dx_1+\ldots.+(n)^1dx_n,$$
$$L_2=A_1^2dx_1+\ldots.+A_n^2dx_n=(1)^2dx_1+\ldots.+(n)^2dx_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$L_p=A_1^pdx_1+\ldots.+A_n^pdx_n=(1)^pdx_1+\ldots.+(n)^pdx_n.$$

The determinants of the matrix of the right-hand members of the system give all the terms of this product since products in the same row could not enter, and those from the same column give zero on integration. Further, the coefficients of $dx_{\alpha_1}\ldots\ldots dx_{\alpha_p}$ is $\Sigma\pm(1)^1(2)^2\ldots\ldots(p)^p$, where the summation means all the permutations of $\alpha_1\ldots\ldots\alpha_p$, and the upper or lower sign is to be taken according as it requires an even or an odd number of changes to bring an element of this sum to the order $\alpha_1\ldots\ldots\alpha_p$. Consequently, remembering the properties of $(1\ldots\ldots p)$ previously defined, we have

$$\Sigma\pm(1)^1(2)^2\ldots\ldots(p)^p=(1\ldots\ldots p).$$

COROLLARY 1. *If the L_i are not linearly independent the resulting I_p is identically zero.*

COROLLARY 2. *Since the determinants may be expanded with respect to their minors of any order we have*

$$(1\ldots\ldots p)(p+1\ldots\ldots p+q)=(1\ldots\ldots p+q)$$

which gives symbolically $I_{p+q}=I_pI_q$.

COROLLARY 3. *When p is odd $(I_p)^2\equiv 0$.*

COROLLARY 4. *The number of coefficients in an I_p is $\dfrac{n!}{p!(n-p)!}$, the same as the number of terms in the product $L_1L_2\ldots\ldots L_p$, consequently, when n independent linear invariants are known all the I_p can be written.*

The converse of this last corollary is not true. For example, the invariant of the second order $I_1^1I_1^2+I_1^3I_1^4$ is not factorable.

THEOREM III. *Knowing an I_{p+1}, an I_p can be found by means of the formula*

$$(1\ldots\ldots p)=\Sigma(1\ldots\ldots p,\,\sigma)X_\sigma.$$

To verify this formula substitute it in equation (2). We get

$$\begin{aligned}
\Sigma[X(1\ldots\ldots p,\sigma)\cdot X_\sigma+(1\ldots\ldots p,\sigma)X(X_\sigma),&\quad \sigma\neq\alpha_1\ldots\ldots\alpha_p,\\
+\sum_\lambda[\lambda,p]\cdot\sum_\sigma(1\ldots\ldots p-1,\lambda,\sigma)X_\sigma,&\quad \sigma\neq\alpha_1\ldots\ldots\alpha_{p-1},\lambda,\\
+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&\\
+\sum_\lambda[\lambda,1]\cdot\sum_\sigma(\lambda,2\ldots\ldots p,\sigma)X_\sigma=0],&\quad \sigma\neq\lambda,\alpha_2\ldots\ldots\alpha_p.
\end{aligned}$$

The subscripts given by $(1,2,\ldots\ldots)$ must be chosen so that there are no repetitions. The coefficient of X_ρ, where $\rho\neq\alpha_1\ldots\ldots\alpha_p$ is

$$\begin{aligned}
X(1\ldots\ldots p,\rho)&+\sum_\sigma(1\ldots\ldots p,\sigma)[\sigma,\rho]\\
&+\sum_\lambda(1\ldots\ldots p-1,\lambda,\rho)[\lambda,p]+\ldots\ldots+\sum_\lambda(\lambda,2\ldots\ldots p,\rho)[\lambda,1]=0.
\end{aligned}$$

These are the conditions for an I_{p+1}. The remaining terms are zero, for we have as coefficients of X_{α_i}, $(i=1, 2 \ldots . p)$

$$\sum_{\sigma}(1 \ldots . p, \sigma)[\sigma, i]+\sum_{\lambda}(1 \ldots . i-1, \lambda, i+1 \ldots . p, i)[\lambda, i].$$

Here the coefficient of any $[\sigma, i]$ is identically zero since interchanging λ and α_i is accomplished by $j+(j-1)$ moves. This number is odd.

The application of this theorem to the problem of finding invariants of order $p-1$ from those of order $p+1$ leads to coefficients identically zero. We have

$$(1 \ldots . p)=\Sigma(1 \ldots . p, \lambda) X_{\lambda}, \qquad \lambda \neq \alpha_1 \ldots . \alpha_p,$$

also

$$(1 \ldots . p-1)=\Sigma(1 \ldots . p-1, \sigma) X_{\sigma}, \qquad \sigma \neq \alpha_1 \ldots . \alpha_{p-1}.$$

Substituting the first equation in the second it becomes

$$(1 \ldots . p-1)=\sum_{\sigma} X_{\sigma} \sum_{\lambda}(1 \ldots . p-1, \sigma, \lambda) X_{\lambda}, \qquad \lambda \neq \sigma.$$

The coefficient of $X_{\sigma} X_{\lambda}$ is

$$(1 \ldots . p-1, \sigma, \lambda)+(1 \ldots . p-1, \lambda, \sigma) \equiv 0.$$

If the equations are canonical then

$$I_2=\int dx_1 dx_2+dx_3 dx_4+\ldots .$$

as can be verified readily. Applying Theorem III, we have

$$A_1=X_2, \quad A_2=-X_1, \ldots .,$$

$$I_1=\int-\frac{\partial F}{\partial x_1} dx_1-\frac{\partial F}{\partial x_2} dx_2-\ldots .-\frac{\partial F}{\partial x_n} dx_n .$$

The condition for the invariant of the *n-th* order is

$$X(1 \ldots . n)+(1 \ldots . n)\left[\frac{\partial X_1}{\partial x_1}+\ldots .+\frac{\partial X_n}{\partial x_n}\right]=0,$$

which becomes

$$\Sigma \frac{\partial}{\partial x_i}\{(1 \ldots . n) X_i\}=0,$$

showing that $(1 \ldots . n)$ is a multiplier of (1). In particular, if the equations are canonical,

$$\Sigma \frac{\partial X_i}{\partial x_i}=0, \text{ and } (1 \ldots . n)=\text{const.}=1$$

is a multiplier.

The invariant of order $n-1$ derived from this is identically zero.

§ 2. *Invariants of the Type* $\int\sqrt[p]{\Sigma(1\ldots.p)dx_{\alpha_1}\ldots.dx_{\alpha_p}}$.

Let

$$I_p=\int\sqrt[p]{\sum_{\alpha_1,\ldots,\alpha_p}(i\ldots.p)dx_{\alpha_1}\ldots.dx_{\alpha_p}}=\int\sqrt[p]{A}.$$

We have

$$X(A^{\frac{1}{p}})=\frac{1}{p}\cdot A^{\frac{1}{p}-1}X(A).$$

Then I_p will be an integral invariant when

$$X(A)=0.$$

This condition, when expanded, is identical with the condition (2). However, in the present case, $(1\ldots.p)$ has the same value for all permutations of $\alpha_1\ldots.\alpha_p$, and may be different from zero when $\alpha_i=\alpha_j$ in the subscript. The conditions to be satisfied are therefore

$$\Sigma[X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]]=0,$$

where λ takes all values 1 to n.

THEOREM IV. *An* I_{p+q} *can be constructed from an* I_p *and an* I_q *by means of the formula*

$$(1\ldots.p+q)=(1\ldots.p)(p+1\ldots.p+q).$$

Substituting this expression in the conditions for an I_{p+q}, namely,

$$X(1\ldots.p+q)+(1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(\lambda,2\ldots.p+q)[\lambda,1]=0,$$

it becomes

$$(1\ldots.p)X(p+1\ldots.p+q)+(p+1\ldots.p+q)X(1\ldots.p)$$
$$+(1\ldots.p)(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(1\ldots.p)(\lambda,p+2\ldots.p+q)[\lambda,p+1]$$
$$+(1\ldots.p-1,\lambda)(p+1\ldots.p+q)[\lambda,p]$$
$$+\ldots.+(\lambda,2\ldots.p)(p+1\ldots.p+q)[\lambda,1]=0,$$

and this condition is satisfied since

$$X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]=0$$

and

$$X(p+1\ldots.p+q)+(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(\lambda,p+2\ldots.p+q)[\lambda,p+1]=0.$$

COROLLARY. *If*

$$I_1^1=\int L_1,\quad I_1^2=\int L_2,\ \ldots.,\quad I_1^p=\int L_p$$

are p linear integral invariants of (1) *then*

$$\int\sqrt[p]{L_1L_2\ldots.L_p}$$

is an integral invariant.

§ 3. *Relative Invariants.*

The following definitions and theorems are due to Poincaré:

(a). An I_p extended to a closed multiplicity E_p in a space of n dimensions can be replaced by an I_{p+1} extended to a multiplicity E_{p+1} in a space of n dimensions limited by the E_p. The coefficients of the I_{p+1} are found from those of the I_p by means of the relations

$$(1\ldots\ldots p+1)=\frac{\partial(1\ldots\ldots p)}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\ldots\ldots p,\,p+1)}{\partial x_{a_1}}+\ldots\ldots\pm\frac{\partial(p+1,1\ldots\ldots p-1)}{\partial x_{a_p}},$$

the upper or lower signs being taken according as p is even or odd.

(b). The expression under the integral sign of I_p is an exact differential when the $(1\ldots\ldots p+1)$ are zero.

(c). The I_{p+1} of (a) is an integral of an exact total differential. This is the generalized Stokes' theorem.

Let $A'_{a_1\ldots\ldots a_p}\equiv(1\ldots\ldots p)'$ represent the left-hand member of equation (2). The increment of I_p was found to be

$$[\textstyle\int^p \sum\limits_{a_1\ldots\ldots a_p}(1\ldots\ldots p)'dx_{a_1}\ldots\ldots dx_{a_p}]\,dt.$$

If

$$\Sigma(1\ldots\ldots p)'dx_{a_1}\ldots\ldots dx_{a_p}$$

is exact, and if its integral be extended to a closed E_p, the resulting equivalent I_{p+1} is identically zero. In this case the original I_p is a relative integral invariant of (1). We shall write it J_p. When

$$(1\ldots\ldots p)'=0, \tag{2}$$

we have an absolute invariant I_p; when

$$\Sigma(1\ldots\ldots p)'dx_{a_1}\ldots\ldots dx_{a_p}\ \textit{is exact}, \tag{6}$$

we have a relative invariant J_p. By means of the equation (5) the statement (6) may be written in full as an equation,

$$\frac{\partial(1\ldots\ldots p)'}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\ldots\ldots p,\,p+1)}{\partial x_{a_1}}+\ldots\ldots\pm\frac{\partial(p+1,\,1\ldots\ldots p-1)}{\partial x_{a_p}}=0. \tag{7}$$

The relation between these two types of invariants may be illustrated by some simple examples.

EXAMPLE 1. Let equations (1) have the canonical form

$$\frac{dx_1}{dt}=\frac{\partial F}{\partial x_2},\quad \frac{dx_2}{dt}=-\frac{\partial F}{\partial x_1},\quad \frac{dx_3}{dt}=\frac{\partial F}{\partial x_4},\quad \frac{dx_4}{dt}=-\frac{\partial F}{\partial x_3}.$$

We have

$$A'_i=\Sigma\left[X(A_i)+A_k\frac{\partial X_k}{\partial x_i}\right]\qquad (i,\,k=1,\,2,\,3,\,4).$$

The integral

$$\int x_2 dx_1 + x_4 dx_3$$

is a relative invariant, since

$$\frac{\partial A'_i}{\partial x_k} - \frac{\partial A'_k}{\partial x_i} = 0.$$

It is not, however, an I_1, since $A'_i \neq 0$.

EXAMPLE 2. Let equations (1) have the canonical form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} = X_i, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} = Y_i \qquad (i=1, 2, \ldots, n).$$

Where F is homogeneous of degree 1 in y_i, or

$$F = \Sigma y_i \frac{\partial F}{\partial y_i} = \Sigma y_i X_i.$$

The partial derivatives of this equation with respect to x_i, y_i are

$$Y_i + \sum_k y_k \frac{\partial X_k}{\partial x_i} = 0, \quad \sum_k y_k \frac{\partial X_i}{\partial y_k} = 0,$$

and these are the necessary and sufficient conditions that

$$I = \int \Sigma y_i dx_i$$

is an absolute invariant.

According to Goursat we designate as D the operation of passing from an I_p to an I_{p+1} by means of equation (5), and as E the operation of passing from an I_p to an I_{p-1} by means of equation (4).

THEOREM V. *The operations D and E are in general not inverse.*

Consider the case of a linear invariant, $I_1 = \int a x_1 + b x_2$ of the linear system of equations in x_1, x_2. By identifying (5) with (4) we have

$$a = \left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_2, \quad b = -\left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_1, \tag{8}$$

where a, b must satisfy the conditions (2); but (2) are satisfied only by $a = \frac{\partial F}{\partial x_1}$, $b = \frac{\partial F}{\partial x_2}$ where F is an integral of the system. With this substitution, however, the second members of (8) become identically zero, consequently, an I_1 satisfying the conditions (8) and (2) does not exist.

§4. *Extension of Poisson's Theorem.*

If equations (1) are in canonical form

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}\qquad (i=1,2,\ldots,n),\tag{1}$$

where F does not contain t explicitly, then $F=$ const. is an integral of (1). Further, if we put

$$[F_1, F_2]\equiv\Sigma\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\frac{\partial F_2}{\partial x_i}\right),$$

we have

$$\frac{dF_1}{dt}=[F_1, F]$$

by means of (1). The necessary and sufficient condition that $F_1=$ const. be an integral of (1) is that $[F_1, F]=0$. Let

$$I_1=\int\Sigma(M_i dx_i+N_i dy_i)$$

be a linear integral invariant of (1), then the conditions to be satisfied are

$$\frac{dM_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial x_k}-N_i\frac{\partial^2 F}{\partial x_i\partial x_k}\right)=0,\tag{m}$$

$$\frac{dN_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial y_k}-N_i\frac{\partial^2 F}{\partial x_i\partial y_k}\right)=0.\tag{n}$$

The coefficients of another similar invariant may be represented by M'_i, N'_i determined by the same conditions and lettered (m′) (n′). These $4n$ equations when multiplied by

$$N'_1,\ldots N'_n,\ -M'_1,\ldots -M'_n,\ -N_1,\ldots -N_n,\ M_1,\ldots M_n,$$

respectively, give on adding,

$$\frac{d}{dt}\Sigma(M_iN'_i-N_iM'_i)=0.$$

Therefore,

$$\Sigma(M_iN'_i-N_iM'_i)=\text{const.}$$

Remembering that

$$M_i=\frac{\partial F_1}{\partial x_i},\quad M'_i=\frac{\partial F_2}{\partial x_i},$$

$$N_i=-\frac{\partial F_1}{\partial y_i},\quad N'_2=-\frac{\partial F_2}{\partial y_i},$$

we have

$$\Sigma\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\frac{\partial F_2}{\partial x_i}\right)\equiv[F_1, F_2]=\text{const}$$

It is well known (theorem of Poisson) that $[F_1, F_2]$ is an integral of (1) when F_1, F_2 are integrals. The above result can be expressed in determinant form

$$\Sigma \begin{vmatrix} M_i & M'_i \\ N_i & N'_i \end{vmatrix} = \text{const.}$$

Consider the case of four known integrals of (1). Then we have

$$\begin{vmatrix} M_i^1 & M_i^2 & M_i^3 & M_i^4 \\ N_i^1 & N_i^2 & N_i^3 & N_i^4 \\ M_k^1 & M_k^2 & M_k^3 & M_k^4 \\ N_k^1 & N_k^2 & N_k^3 & N_k^4 \end{vmatrix} = \text{const.}$$

since all the minors of the second order found in the first and second rows, also in the third and fourth rows, are constant.

And generally

$$\sum_{\alpha_1}\sum_{\alpha_2}\ldots\sum_{\alpha_p} \begin{vmatrix} M^1_{\alpha_1} & M^2_{\alpha_1}\ldots\ldots & M^{2p}_{\alpha_1} \\ N^1_{\alpha_1} & N^2_{\alpha_1}\ldots\ldots & N^{2p}_{\alpha_1} \\ \vdots & & \\ M^1_{\alpha_p} & M^2_{\alpha_p}\ldots\ldots & M^{2p}_{\alpha_p} \\ N^1_{\alpha_p} & N^2_{\alpha_p}\ldots\ldots & N^{2p}_{\alpha_p} \end{vmatrix} \equiv \sum_{\alpha_1\ldots\ldots\alpha_p} \Delta_{\alpha_1\ldots\ldots\alpha_p} = \text{const.}$$

We may assume all the α_i different in making the summation, because otherwise the particular determinant involved would be zero.

In particular if $p=n$ we have, since the total number of determinants in the sum is $n!$, and, since the permutation of the numbers $\alpha_1\ldots\ldots\alpha_p$ corresponds to a transposition of the rows keeping rows $2k-1$, $2k$ together,

$$\sum_{\alpha_1\ldots\ldots\alpha_n} \Delta_{\alpha_1\ldots\ldots\alpha_n} = n!\,\Delta_{\alpha_1\ldots\ldots\alpha_n} = n!\,\Delta_{1,2\ldots\ldots n}.$$

$\Delta_{1\ldots\ldots n}$ is simply the functional determinant of the F_i with respect to the x_i, y_i. This is the extension of Poisson's theorem.

§ 5. *Least Action.*

Consider the canonical equations of dynamics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which admit the relative invariant

$$J_1 = \int \Sigma p_i dq_i,$$

and the equivalent absolute invariant (by Stokes' theorem)

$$I_2 = \int\int \Sigma dp_i dq_i.$$

Goursat* has shown that the integral invariant of *p-th* order

$$I_p = \int_{E_p} \Sigma A_{\alpha_1 \ldots \alpha_p} dx_{\alpha_1} \ldots dx_{\alpha_p}$$

of the system

$$\frac{dx_i}{dt} = X_i \qquad (i=1, 2, \ldots, n)$$

is identical with

$$I_p = \int_0^T dt \int_{E_{p-1}} \Sigma C_{\alpha_1 \ldots \alpha_{p-1}} dx_{\alpha_1} \ldots dx_{\alpha_{p-1}},$$

where

$$C_{\alpha_1 \ldots \alpha_{p-1}} = \sum_{i=1}^{n} A_{\alpha_1 \ldots \alpha_{p-1}, i} X_i .$$

By specializing this result so as to apply to the above canonical system and its I_2 we have

$$I_2 = \int dt \int -dH.$$

Now, if this integral be extended to any E_2 for which $H=$ const., we have

$$I_2 = 0, \quad J_1 = 0.$$

It is to be noted that in this transformation of I_2 the form has been simplified *by means of the differential equations.* We can now prove the following:

THEOREM VI. *The relative invariant J_1 represents twice the action in a dynamical system, that is, $\int T dt$.*

We have $H = T - U$ where T is independent of the q_i and homogeneous of the second degree in the p_i, and U is a function of the q_i only. Then we have

$$J_1 = \int \Sigma p_i dq_i = \int \Sigma p_i \frac{\partial H}{\partial p_i} dt = \int 2T dt.$$

UNIVERSITY OF ILLINOIS, *September*, 1916.

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* Goursat, *Journal de Mathématiques pures et appliquées*, 7 série, Tome I (1915).

Fundamental Regions for Certain Finite Groups in S_4.

By Henry F. Price.

One of the most interesting results of the study of transformations is what Klein has termed the "fundamental region."

A fundamental region for a group of transformations is a system of points which contains one and only one point of every conjugate set.

Fundamental regions in the complex plane have been studied for some time and are well known. Klein * and his followers have developed the subject to a considerable extent. There is a close relationship between this subject and the elliptic modular functions and the reduction of quadratic forms.

The fundamental regions for groups in more than one complex variable have not been studied much. However, J. W. Young,† in a recent paper, obtained such regions for cyclic groups in two complex variables.

In this paper will be considered fundamental regions for certain finite groups in two complex variables. The octahedral and icosahedral groups will be dealt with.

The fundamental regions for these groups in the real plane can be readily determined and found to be triangles bounded by the axes of reflections. In the case of the complex plane the problem is solved by using Hermitian forms which meet the real plane in the sides of these triangles.

The problem will be solved completely in the case of the octahedral group. In the case of the icosahedral group it will be solved except for the points which reduce one or more of the Hermitian forms to zero.

The ternary collineation group G_{24} can be generated by the following three operations: $E_1[-\xi_1, \xi_3, \xi_2]$, $E_2[\xi_2, \xi_1, \xi_3]$ and $E_3[\xi_1, \xi_3, \xi_2]$.

It permutes the points of the real plane. As it contains nine operations of order 2, there are nine reflections. As the group is simply isomorphic with the symmetric group on four letters, it is evident that these reflections are in

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* Felix Klein, "Elliptischen Modulfunktionen, Vol. I, pp. 183–207.

† J. W. Young, "Fundamental Regions for Cyclical Groups of Linear Fractional Transformations on Two Complex Variables," *Bull. Amer. Math. Soc.*, Vol. XVII, p. 340.

two conjugate sets. The axes of the reflections, $\xi_1=\pm\xi_3$, $\xi_2=\pm\xi_3$, $\xi_1=\pm\xi_2$, $\xi_1=0$, $\xi_2=0$ and $\xi_3=0$, divide the plane into twenty-four triangles.

Any point in one of these triangles can be transformed, by a suitably chosen operation of the group, into a point in any other triangle. Any triangle is then a *fundamental region* in the plane for the group G_{24}.

The group permutes the complex points of the plane also. If we consider the totality of points in the plane, complex as well as real, as real points in four-space, we may ask the question whether fundamental regions exist in S_4 for the group under consideration. If one does exist it must contain one triangle, and only one, of the real plane. The fixed points of the transformations would lie on the boundaries of such a fundamental region.

Consider the Hermitian forms $\xi_1\bar{\xi}_1-\xi_2\bar{\xi}_2$ and $\xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2$ in which $\dfrac{\xi_1}{\xi_3}=x+iu$ and $\dfrac{\xi_2}{\xi_3}=y+iv$. Under the G_{24} we have two conjugate sets of three forms each:

$$
\begin{array}{llll}
(1) & \xi_1\bar{\xi}_1-\xi_2\bar{\xi}_2, & & (4)\ \ \xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2, \\
(2) & \xi_2\bar{\xi}_2-\xi_3\bar{\xi}_3, & \text{and} & (5)\ \ \xi_2\bar{\xi}_3+\bar{\xi}_2\xi_3, \\
(3) & \xi_3\bar{\xi}_3-\xi_1\bar{\xi}_1, & & (6)\ \ \xi_3\bar{\xi}_1+\bar{\xi}_3\xi_1.
\end{array}
$$

It is evident that there is at least one relation between the forms, *i. e.*, $(1)+(2)+(3)\equiv 0$.

If we consider the portion of S_4 in which the signs of (1), (3), (4), (5) and (6) are all $+$ and make use of the relation $(1)+(2)+(3)\equiv 0$, we see that the sign of (2) is determined as $-$.

This region will be written $[+-+,\ +++]$ where the signs of the six forms are written in order.

We shall next consider into how many such regions S_4 is divided by the six Hermitian forms.

The forms (1), (2) and (3) are conjugate under G_{24}, and because of the relation $(1)+(2)+(3)\equiv 0$ admit at most six arrangements of sign. The forms (4), (5) and (6) are also conjugate under G_{24}, and admit at most eight arrangements of sign. There are therefore forty-eight possible choices of sign for (1), (2) .... (6). But the eight arrangements divide into two complete conjugate sets of four under G_{24} according as the number of $+$ signs is odd or even. For one of such four choices for the forms (4), (5), (6), the six arrangements of sign for (1), (2), (3) all are conjugate, *e. g.*, the $+++$ choice is unaltered by the group G_6 of permutations of the variables, and the G_6 permutes all arrangements for (1), (2), (3).

Hence the forty-eight possible choices of sign or possible regions in S_4 divide into two conjugate sets of twenty-four each, and two of these regions, one from each set, constitute a fundamental region in S_4 for G_{24}.

The region $[+-+, +++]$ belongs to one of the sets. By changing the sign of (5) to $-$ we obtain a region $[+-+, +-+]$ of the other set.

Taking these two regions together we obtain $\Gamma=[+-+, +\pm+]$ which is a fundamental region in S_4 for the group G_{24}, except for the points which reduce one or more of the Hermitian forms to zero.

If we consider the points which reduce one or more of the forms to zero we are dealing with what we may call the "boundaries" of the fundamental regions.

By placing the six forms in Γ equal to zero singly, in pairs, in groups of three, etc., in all possible ways, and discarding those which are conjugates of others, it is found that there are twenty-three sets of points which are sections of Γ's boundaries and which should be taken in the fundamental region for the group. Γ can be defined completely, therefore, by the sets of points:

$$[+-+,\ +\pm+],$$

$$\begin{array}{llll}
[+-+,\ +0+], & [0-+,\ +\pm+], & [+-0,\ +\pm+], & [+-+,\ 0++], \\
[+-+,\ ++0], & [0-+,\ +0+], & [+-0,\ +0+], & [+-+,\ 000], \\
[+-+,\ 0+0], & [0-+,\ 0++], & [+-0,\ ++0], & [000,\ +\pm+], \\
[+-+,\ 00+], & [0-+,\ 00+], & [+-0,\ 00+], & [000,\ +0+], \\
[+-+,\ +00], & [0-+,\ +00], & [+-0,\ 0+0], & [000,\ 00+], \\
 & [0-+,\ 000], & [+-0,\ 000]. &
\end{array}$$

The ternary collineation group G_{60} furnishes a more complex fundamental region in S_4 than G_{24} does. It is well known that this group can be generated by the three operations:

$$E_1[\xi_2, \xi_3, \xi_1]; \quad E_2[\xi_1, -\xi_2, -\xi_3];$$

and

$$E_3\begin{cases}\xi_1'= \xi_1-\alpha\xi_2+(\alpha+1)\xi_3, \\ \xi_2'=-\alpha\xi_1+(\alpha+1)\xi_2+\xi_3, \\ \xi_3'= (\alpha+1)\xi_1+\xi_2-\alpha\xi_3,\end{cases}$$

where $\alpha=\dfrac{-1\pm\sqrt{5}}{2}$.*

This group permutes the points of the real plane. As it contains fifteen operations of order 2, there are fifteen axes of reflections. These lines divide

* H. H. Mitchell, "Determination of the Ordinary and Modular Ternary Linear Groups," *Trans. Amer. Math. Soc.*, Vol. XII, No. 2, p. 223.

the plane into sixty triangles. The intersections of these fixed lines are *real* fixed points of three classes; first, the points left invariant under the fifteen subgroups of order 4; second, the points invariant under the ten subgroups of order 6; and third, the points invariant under the six subgroups of order 10.

Each of the sixty triangles into which the plane is divided by the fifteen fixed lines has for its vertices one of each of the three classes of fixed points.

Each of the sixty triangles is a fundamental region in the plane. The group also permutes the complex points of the plane. Just as in the case of the G_{24} we can consider the totality of real and complex points in the plane as real points in S_4 and seek a fundamental region for the group G_{60} in the higher space.

Consider the Hermitian form $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$. Under G_{60} there is a single set of fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$, which can be expressed in terms of six forms:

$$
\begin{aligned}
F_1 &= \quad (\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3+3\xi_1\bar{\xi}_2+3\bar{\xi}_1\xi_2,\\
F_2 &= \quad (\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3-3\xi_3\bar{\xi}_1-3\bar{\xi}_3\xi_1,\\
F_3 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3+3\xi_2\bar{\xi}_3+3\bar{\xi}_2\xi_3,\\
F_4 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3-3\xi_2\bar{\xi}_3-3\bar{\xi}_2\xi_3,\\
F_5 &= \quad (\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3+3\xi_3\bar{\xi}_1+3\bar{\xi}_3\xi_1,\\
F_6 &= \quad (\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3-3\xi_1\bar{\xi}_2-3\bar{\xi}_1\xi_2.
\end{aligned}
$$

The fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$ are $F_{ik}=-F_{ki}=F_i-F_k$. There are twenty relations between these forms $F_{ij}+F_{jk}+F_{ki}\equiv 0$.

It is found that the Hermitian forms F_{12}, F_{54}, F_{16}, F_{34} and F_{52} meet the real plane in the sides of the triangle whose sides are $x+\alpha y-\alpha^2=0$, $x=0$ and $y=0$.

For any point which lies in this triangle, the signs of F_{12} and F_{54} are both $-$, while those of F_{16}, F_{34} and F_{52} are all $+$.

Taking F_{12} and F_{54} as negative and the other three forms as positive, and making use of the twenty relations between the forms, it is seen that the signs of the remaining ten forms are determined. Therefore, none of these ten forms can *cross* the *region* in S_4 which is determined by the five forms under consideration. Since for this region $F_3>F_4>F_5>F_2>F_1>F_6$, it can be written [345216].

Under G_{60} the region [345216] has sixty conjugate regions. These meet the real plane in distinct triangles, the fundamental regions, for the group, in the plane.

The question arises into how many such regions is S_4 divided by the forms F_{ik}?

For any point in S_4 not on an F_{ik} the values of the F_i are all distinct, and, since these six values admit at most seven hundred and twenty permutations, the forms F_{ik} have at most seven hundred and twenty arrangements of sign.

Under G_{60} the seven hundred and twenty value systems of the F_i divide into twelve conjugate sets of sixty each, and if one value system is taken from each set we obtain a fundamental region. As an example of such a fundamental region we give the twelve value systems determined by the inequalities $F_3 > F_4 > F_1$, $F_4 > F_6$, $F_5 > F_2 > F_1$ and $F_2 > F_6$. No two of these are conjugates under G_{60} and therefore they comprise a fundamental region in S_4 for the group, except for the points which reduce one or more of the F_{ik} to zero.